Infinite Behaviour and Fairness in Petri Nets.

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INFINITE BEHAVIOUR AND FAIRNESS IN PETRI NETS

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Abstract

Several classes of ω-languages of labelled Petri nets are defined and related to each other. Since such nets can be interpreted to behave fair, these notions are compared with explicit definitions of fairness for nets.

1. Introduction

Concurrent systems often appear as discrete time continuous dynamical processes. Therefore it is natural to represent the behaviour of a Petri net by the set of all infinite sequences of transitions that can occur. Infinite sequences have been used to study traditional problems of deadlocks and starvation [Nivat].

The subject of this paper is the notion of fairness in Petri nets. There are many opinions about the meaning of fairness in concurrent systems. In general, fairness expresses the observance of certain rules for the actors.

For Petri nets we first give such rules as specifications on markings and in the same manner for transitions. By this only those sequences are allowed (behave fair) which repeat some markings (or transitions) infinitely often. As an example we mention Dijkstra's problem of the dining five philosophers. They will behave fair, if every philosopher is infinitely often in a state (marking), in which he is eating. Another approach to fairness in Petri nets is to demand that the selection of transitions, enabled at the same time, will be "fair".

For the first approach we adopt definitions for ω-languages, which were developed by [Landweber] for ω-languages of finite automata. He introduced some definitions of the acceptance of infinite sequences and called them i-successful for $i \in \{1,1',2,2',3\}$. To give an example, 3-successful means that exactly every state of a designated "anchor set" is visited infinitely often. Investigations of such i-successful ω-languages were also made for push down automata and turing machines by [Cohen, Gold] and [Hossley].

For Petri nets we define i-successful ω-languages in three different ways. First we consider markings of the net as anchor states, then only markings on some bounded places, and at last the fired transitions. The three behaviours of Petri nets are called i-behaviour, bounded i-behaviour, and transitional i-behaviour.

In sections three and four we define thoses classes of i-accepting languages for nets and compare the expressional power of these classes with each other.

The second approach to fairness in Petri nets is that every transition has a kind of finite delay property. This means that the firing rule is extended so that no transition is infinitely long (or often) enabled without firing. In section five we will define ω-languages of Petri nets with such firing rules and compare their expressional power with the clases of languages investigated before.

In this paper we have not drawn out all proofs perfectly. Most of them are described in [Valk] and [Carstensen1] in detail.

2. Basic Definitions and Notations

This section recalls some definitions about ω-languages, infinite words and Petri net languages. It defines also the infinite behaviour of Petri nets.

a) Basic definitions on ω-languages:

The symbol for *infinitely many* is $\omega := |\mathbb{N}|$ (the cardinality of the nonnegative integers). The arithmetic operations on $\mathbb{N}$ are extended to $\mathbb{N} \cup \{\omega\}$ by $\forall n \in \mathbb{N}$: $n < \omega$ and $n + \omega = \omega + n = \omega + \omega = \omega$.

The quantor *there are infinitely many* is defined by

$\exists_{\infty} x : P(x) :\Leftrightarrow |\{x \mid P(x)\}| = \omega$.

We now give some definitions of languages. Let X be always an arbitrary but finite set (alphabet).

A mapping $v: M \to X$, where $M = \{1,\ldots,k\}$ $(k \in \mathbb{N}^+)$ or $M = \mathbb{N}^+$, is called *sequence* over X, v(i) denotes the i-th element ($i \in \mathbb{N}$) of v. The cardinality of M is the *length* of v, i.e. $|v| := |M|$. If $|v| \neq \omega$ v is called *finite* otherwise *infinite* sequence. The set of all finite sequences over X is denoted by X^*, the set of all infinite sequences by X^ω , and $X^\infty := X^* \cup X^\omega$. We often use the notation *word* instead of sequence for elements of languages.

For a sequence v over X , v[i] denotes the *prefix* with the length i of v , for $i \leq |v|$, $v[i] : \{1,\ldots,i\} \to X$ with $v[i](j) = v(j)$ for all $j \leq i$. v[0] is defined as the empty word λ.

For an infinite sequence $v \in X^\omega$ we define the *infinity set* of v:

$In(v) := \{x \in X \mid \exists_{\infty}\, i \in \mathbb{N}^+ : v(i) = x\}$.

b) Basic definitions on Petri nets and their ω-behaviour:

A sixtupel $N = (S,T,F,W,h,m_o)$ is called *λ-free (labelled) Petri net*, if (S,T,F) is a directed net, the set of places S and the set of transitions T are finite and disjoint sets, $F \subseteq (S \times T) \cup (T \times S)$ is called the flow relation; $W: F \to \mathbb{N}^+$

is the multiplicity function, $h: T \to X$ the labelling function, and $m_O: S \to \mathbb{N}$ the initial marking.
The well-known firing rule and token game are assumed, for more details see [Genrich, Stankiewics-Wichno].
Beside this definition we will also use a vector representation of the flow relation and the markings.
For a Petri net we define the *backward incidence matrix* $U: SxT \to \mathbb{N}$ by

$U(s,t) = \begin{cases} W(s,t) & \text{if } (s,t) \in F \\ 0 & \text{otherwise} \end{cases}$, the *foreward incidence matrix* $V: SxT \to \mathbb{N}$ by

$V(s,t) = \begin{cases} W(t,s) & \text{if } (t,s) \in F \\ 0 & \text{otherwise} \end{cases}$, and the *incidence matrix* $C: SxT \to Z$ by

$C := V - U$. A marking is treated as a one-column-matrix (or a vector) and a column of the matrix C is denoted by $C(-,t)$.

A sequence of transitions is called *firing sequence* of N, if the successively firing of the transitions is allowed by the firing rule; an infinite sequence of transitions is an *infinite firing sequence* if every prefix is a firing sequence. The set of all (infinite) firing sequences of N is denoted by $F(N)$ ($F_\omega(N)$). The sets $F(N)$ and $F_\omega(N)$ are languages over the transitions, the corresponding classes of languages are $\mathcal{F}$ and $\mathcal{F}_\omega$, respectively.
For a Petri net N we call $M(N) := \mathbb{N}^{|S|}$ the marking set of N. Let $v \in T^*$ ($v \in T^\omega$) be a firing sequence, we call $\delta_0(v): \{0,\ldots,|v|\} \to M(N)$ ($\mathbb{N} \to M(N)$) the *marking sequence* of v, if
$\forall\, i \in \mathbb{N}, i \leq |v| : \delta_0(v)(i) = m_O + \sum_{j=1}^{i} C(-,v(j))$. The set of all *reachable markings* from m_O is defined as $(m_O\rangle := \{m \mid \exists\, v \in F(N)\ \exists\, i \in \{1,\ldots,|v|\} : \delta_0(v)(i) = m\}$.

We extend the labelling function $h: T \to X$ to $h: T^\infty \to X^\infty$ by $h(v)(i) = h(v(i))$ for all $v \in T^\infty$ and $i \leq |v|$. Now we define the *language* (or the *behaviour*) of a labelled Petri net by
$L(N) := \{h(v) \in X^* \mid v \in F(N)\}$ and $L_\omega(N) := \{h(v) \in X^\omega \mid v \in F_\omega(N)\}$.
The corresponding classes of languages are denoted by $\mathcal{L}$ and $\mathcal{L}_\omega$, respectively.
Let $\mathcal{L}_a$ and $\mathcal{L}_b$ be classes of languages of finite words and $\mathcal{L}_c$ a class of ω-languages. Then we define:

For $L \in \mathcal{L}_a$, $L^\omega := \{ w \in X^\omega \mid w = \prod_{i=1}^{\infty} w_i \text{ and } \forall\, i \in \mathbb{N}^+ : w_i \in L\}$;

$\mathcal{L}_a \circ \mathcal{L}_c := \{ \bigcup_{i=1}^{k} A_i B_i \mid A_i \in \mathcal{L}_a, B_i \in \mathcal{L}_c, k \in \mathbb{N}\}$;

$\mathcal{L}_a \overset{\circ}{\omega} \mathcal{L}_b := \{ \bigcup_{i=1}^{k} A_i (B_i)^\omega \mid A_i \in \mathcal{L}_a, B_i \in \mathcal{L}_b, k \in \mathbb{N}\}$; and

$KC_\omega(\mathcal{L}_a) := \mathcal{L}_a \overset{\circ}{\omega} \mathcal{L}_a$ (the ω-Kleene-closure of $\mathcal{L}_a$).

Let N be a labelled Petri net and $D \subseteq M(N)$ a finite set of markings, then $L(N,D) := \{h(v) \in X^* \mid v \in F(N) \text{ and } \delta_0(v)(|v|) \in D\}$ is the *terminal language* of (N,D) and the *cyclic language* of N is $L_{cyc}(N) := L(N,\{m_o\})$. The corresponding classes of languages are denoted by $\mathcal{L}_0$ and $\mathcal{L}_{cyc}$.

3. Specification of Fairness by Anchor Sets of Markings

Finite languages of finite automata or push-down automata are defined by all possible sequences reaching certain final states. For ω-languages this definition makes no sense. In the theory of ω-languages it is usual to mark sets of states so that all sequences allowed must hold certain conditions with respect to these states. Such conditions were introduced by Landweber and the accepted infinite sequences were called i-successful [Landweber]. (Originally this definition was made for $i \in \{1,1',2,2',3\}$, we extend it to the case of $i = 3'$).

We will transform these definitions from automata to Petri nets. In the same manner as terminal languages of nets were defined, we will consider all possible markings of a net as its set of states in this section.

Let Y be a finite or infinite set, $\mathcal{A} \subseteq \mathcal{P}(Y)$ a finite set of finite non-empty subsets of Y, and $u \in Y^{\omega}$ an infinite sequence over Y. u is called

1 -successful or *touching* for $\mathcal{A}$, if	$\exists A \in \mathcal{A}\ \exists i \in \mathbb{N}^+ : u(i) \in A$
1'-successful or *completely enclosed* for $\mathcal{A}$, if	$\exists A \in \mathcal{A}\ \forall i \in \mathbb{N}^+ : u(i) \in A$
2 -successful or *repeatedly successful* for $\mathcal{A}$, if	$\exists A \in \mathcal{A} : \emptyset \neq In(u) \cap A$
2'-successful or *eventually enclosed* for $\mathcal{A}$, if	$\exists A \in \mathcal{A} : \emptyset \neq In(u) \subseteq A$
3 -successful or *eventually terminal* for $\mathcal{A}$, if	$\exists A \in \mathcal{A} : In(u) = A$
3'-successful or *continual* for $\mathcal{A}$, if	$\exists A \in \mathcal{A} : A \subseteq In(u)$.

Let N be a λ-free Petri net, $\mathcal{D} \subseteq \mathcal{P}(M(N))$ a finite set of finite non-empty sets of markings, then we define the *i-behaviour* of $(N,\mathcal{D})$ for $i \in \{1,1',2,2',3,3'\}$ by
$L^i_{\omega}(N,\mathcal{D}) := \{h(v) \mid v \in F_{\omega}(N)$ and $\delta_0(v)$ is i-successful for $\mathcal{D}\}$.
The corresponding classes of all such i-behaviours are denoted by $\mathcal{L}^i_{\omega}$.
In this definition we introduce some restrictions for the empty set which are usually not considered. The definitions were made for infinite words in finite automata, so that every word must visit at least one state infinitely often. In a Petri net this must not be the case, e.g. a net which counts the number of firings on one place never reaches a marking twice. For this reason we only allow non-empty sets to be member of $\mathcal{A}$ and $\emptyset \neq In(u)$ for 2'-successful. In our papers [Valk] and [Carstensen 1] we had this restriction in mind but forgot to mention it.

To improve the understanding of these definitions and their relations to fairness the reader is invited to look at the example at the end of section 4.

In the same manner as we defined i-successful words for Petri nets, it was done for finite automata and push-down automata. The resulting classes of ω-languages are denoted by $\mathcal{R}^i_{\omega}$ and $\mathcal{PD}^i_{\omega}$, respectively.

Observation: $\mathcal{R}_{\omega}^{i} \subseteq \mathcal{L}_{\omega}^{i}$ (since every finite automaton can be seen as a labelled Petri net)

Lemma 3.1: Each class $\mathcal{L}_{\omega}^{i}$, for $i \in \{1,1',2,2',3,3'\}$, is closed under finite union.

The proof of this lemma is very similar to the corresponding proofs for final net languages [Hack]. We will omit it in this paper.

Since the closure under finite union holds for all classes of ω-languages mentioned in this paper (with the exception of $\mathcal{F}_{\omega}$), the inclusion between an i-successful language and an ω-language must only be proved for behaviours with a singleton anchor set $\mathcal{A} = \{A\}$ instead of the general case $\mathcal{A} = \{A_1,\ldots,A_K\}$.

For a labelled Petri net $N = (S,T,F,W,h,m_o)$ and a marking $d \in M(N)$ we define N_d to be the net N with the new initial marking d , i.e. $N_d := (S,T,F,W,h,d)$.

Theorem 3.1:
a) $\mathcal{L}_{\omega}^{1} = \mathcal{L}_{o} \circ \mathcal{L}_{\omega}$,
b) $\mathcal{L}_{\omega}^{1'} = \mathcal{R}_{\omega}^{1'}$,
c) $\mathcal{L}_{\omega}^{2} = \mathcal{L}_{o} \overset{o}{\omega} \mathcal{L}_{cyc} \subseteq \mathcal{L}_{\omega}^{3'}$,
d) $\mathcal{L}_{\omega}^{2'} = \mathcal{L}_{o} \circ \mathcal{L}_{\omega}^{1'}$, and
e) $\mathcal{L}_{\omega}^{3} = \mathcal{L}_{o} \circ \mathcal{R}_{\omega}^{3}$.

Proof:

1. Let be $L = L_{\omega}^{i}(N, \{D\})$.
For the case $i = 1'$ there are only finitely many markings allowed, thus the behaviour can also be realized by a finite automaton, i.e. the marking graph only with the markings of D .

2. In the other cases every successful firing sequence must consist of two parts. The first part will lead to a marking in D , i.e. a word from a language in $\mathcal{L}_o$. We refer to the definition of i-successful: it is excluded that an anchor set is empty and in the case of $i=2'$ that there is no marking visited infinitely often. In the case of finite automata these restrictions are not necessary, since the set of states if finite. In the case of Petri nets, however, the set of reachable markings is not necessarily finite.

The second part will start in that marking (there are only finitely many) and have the following properties:

For $i = 1$: there is no restriction for the firing sequence.
For $i = 2$: the marking from which it starts must be reached infinitely often.
For $i = 2'$: only markings of D may be used.
For $i = 3$: exactly the markings of D must be reached infinitely often.

3. For the other direction of the inclusions there are similar methods used as in the case of concatenations of net languages.

4. To show $\mathcal{L}_\omega^2 \subseteq \mathcal{L}_\omega^{3'}$:
   Let $L = L_\omega^2(N, \{D\})$. We define $\mathcal{D}' := \{D' \mid \emptyset \neq D' \subseteq D\}$, then
   $L = L_\omega^{3'}(N, \mathcal{D}')$.

We are now able to give a hierarchy of theses classes of languages.

<u>Theorem</u> 3.2:

$$\begin{array}{ccccccccccc} \mathcal{L}_\omega & \subsetneq & \mathcal{L}_\omega^1 & & & & & & & & \\ \cup\!\!\!\!\not\; & & \cup\!\!\!\!\not\; & & & & & & & & \\ \mathcal{L}_\omega^{1'} & \subsetneq & \mathcal{L}_\omega^{2'} & \subsetneq & \mathcal{L}_\omega^3 & \subsetneq & \mathcal{L}_\omega^2 & \subsetneq & \mathcal{L}_\omega^{3'} & \subsetneq & KC_\omega(\mathcal{L}_0) \end{array}$$

<u>Proof:</u> We will show that the inclusions are valid and that these are the only inclusions between the mentioned classes.

a) $\mathcal{L}_\omega \subseteq \mathcal{L}_\omega^1$ since $\mathcal{L}_\omega^1 = \mathcal{L}_o \circ \mathcal{L}_\omega$ and $\{\lambda\} \in \mathcal{L}_o$.

b) $\mathcal{L}_\omega^{1'} \subseteq \mathcal{L}_\omega$ since $\mathcal{L}_\omega^{1'}$ can be seen as the behaviour of a finite automaton and hence as the ω-behaviour of a Petri net.

c) $\mathcal{L}_\omega^{1'} \subseteq \mathcal{L}_\omega^{2'}$ since $\mathcal{L}_\omega^{1'} = \mathcal{R}_\omega^{1'}$, $\mathcal{L}_\omega^{2'} = \mathcal{L}_0 \circ \mathcal{R}_\omega^{1'}$ and $\{\lambda\} \in \mathcal{L}_0$.

d) $\mathcal{L}_\omega^{2'} \subseteq \mathcal{L}_\omega^1$ since $\mathcal{L}_\omega^{2'} = \mathcal{L}_0 \circ \mathcal{R}_\omega^{1'} = \mathcal{L}_0 \circ \mathcal{L}_\omega^{1'} \subseteq \mathcal{L}_0 \circ \mathcal{L}_\omega = \mathcal{L}_\omega^1$.

e) $\mathcal{L}_\omega^{2'} \subseteq \mathcal{L}_\omega^3$ since $\mathcal{L}_\omega^{2'} = \mathcal{L}_0 \circ \mathcal{R}_\omega^{1'} \subseteq \mathcal{L}_0 \circ \mathcal{R}_\omega^3 = \mathcal{L}_\omega^3$.

f) $\mathcal{L}_\omega^3 \subseteq \mathcal{L}_\omega^2$: It holds: $\mathcal{R}_\omega^3 = KC_\omega(\mathcal{R}) = \mathcal{R} \underset{\omega}{\circ} \mathcal{R}$ [Eilenberg]
   and $C \in \mathcal{R} \Rightarrow \exists D \in \mathcal{L}_{cyc} : C^\omega = D^\omega$.

   Let be $L \in \mathcal{L}_\omega^3 = \mathcal{L}_0 \circ \mathcal{R}_\omega^3 = \mathcal{L}_0 \circ (\mathcal{R} \underset{\omega}{\circ} \mathcal{R})$, thus L is a finite union of languages $A_i B_i C_i^\omega$, where $A_i \in \mathcal{L}_0$, B_i, $C_i \in \mathcal{R}$. A terminal net language concatenated with a regular language is also a terminal net language, hence $A_i B_i \in \mathcal{L}_0$. And there is a cyclic net language $D \in \mathcal{L}_{cyc}$ with $C^\omega = D^\omega$, hence $L \in \mathcal{L}_0 \underset{\omega}{\circ} \mathcal{L}_{cyc} = \mathcal{L}_\omega^2$.

g) $\mathcal{L}_\omega^2 \subseteq \mathcal{L}_\omega^{3'}$ (see Theorem 3.1)

h) $\mathcal{L}_\omega^{3'} \subseteq KC_\omega(\mathcal{L}_0)$: Let be $L = L_\omega^{3'}(N, \{D\}) \in \mathcal{L}_\omega^{3'}$, $D = \{d_1, \ldots, d_k\}$, then we can write L as the concatenation of terminal net languages
   $L = L(N,\{d_1\}) \,.\, [L(N_{d_1},\{d_2\}) \,.\, \ldots \,.\, L(N_{d_k},\{d_1\})]^\omega$. The class $\mathcal{L}_0$ is closed under finite concatenation [Hack], thus
   $\exists L2 \in \mathcal{L}_0 : L = L(N,\{d_1\}) \,.\, L2^\omega \in \mathcal{L}_0 \underset{\omega}{\circ} \mathcal{L}_0 = KC_\omega(\mathcal{L}_0)$.

The following parts of the proof show that there are no more inclusions.

i) $\mathcal{L}_\omega^{2'} \not\subseteq \mathcal{L}_\omega$: Let $L = a^* b c^\omega \in \mathcal{R}_\omega^{2'} \subseteq \mathcal{L}_\omega^{2'}$, $L \notin \mathcal{L}_\omega$. Suppose there is a net N so that $L = L_\omega(N)$, then there is a firing sequence u in N with $h(u) = a^\omega$, but $a^\omega \notin L$.

j) $\mathcal{L}_\omega \not\subseteq KC_\omega(\mathcal{L}_0)$: Consider the net in figure 3.1 . N is a net with a language L(N) which has a not semilinear Parikh image. $v \in L(N) \Rightarrow |v|_a < 2^{|v|_d + 1}$
   For the language $L = L_\omega(N)$ it holds that $L \notin KC_\omega(\mathcal{F})$ for any family of languages $\mathcal{F} \subseteq \mathcal{P}(X^*)$.

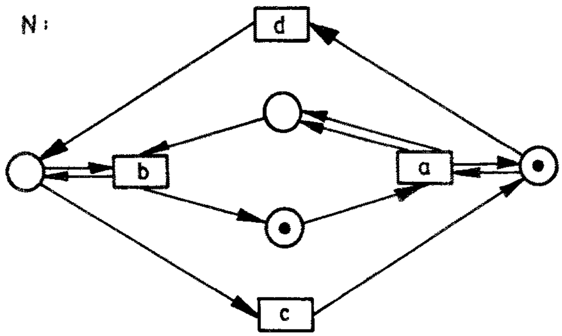


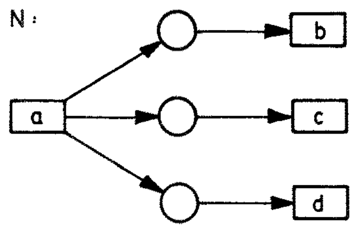
Figure 3.1

k) $\mathcal{L}^3_\omega \not\subseteq \mathcal{L}^1_\omega$: $L = (a^* b)^\omega \in \mathcal{R}^3_\omega \subseteq \mathcal{L}^3_\omega$, but $L \notin \mathcal{L}^1_\omega$.

l) $\mathcal{L}^2_\omega \not\subseteq \mathcal{L}^3_\omega$: Consider the net N:

and $D = \{(\underline{0})\}$. $L^2_\omega(N,\{D\}) \notin \mathcal{L}^3_\omega$.

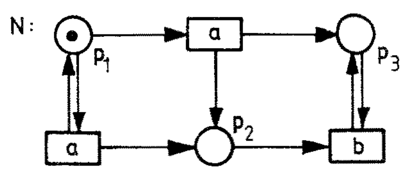
m) $\mathcal{L}^{3'}_\omega \not\subseteq \mathcal{L}^2_\omega$: Consider the net in figure 3.2 . It can be shown that $L^{3'}_\omega(N,\{D\}) \notin \mathcal{L}^2_\omega$.



$$D = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

Figure 3.2

n) $KC_\omega(\mathcal{L}_0) \not\subseteq \mathcal{L}^{3'}_\omega$: Consider the net N in figure 3.3. Let $L = [L(N,D)]^\omega$, i.e. $L = A^\omega$ with $A = \{a^n b^n \mid n \geq 1\}$, then $L \notin \mathcal{L}^{3'}_\omega$. (In [Valk] the proof was made for $\mathcal{L}^2_\omega$, but it remains unchanged for $\mathcal{L}^{3'}_\omega$.)



$$D = \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

Figure 3.3

4. Specifications of Fairness by Anchor Sets of Markings on Bounded Places and of Transitions

The hierarchy of ω-languages $\mathcal{L}^i_\omega$ in theorem 3.2 differs from the corresponding hierarchies of nondeterministic finite automata and of nondeterministic push-down automata. ($\mathcal{P}D^{1'}_\omega \subseteq \mathcal{P}D^1_\omega = \mathcal{P}D^{2'}_\omega \subseteq \mathcal{P}D^2_\omega = \mathcal{P}D^3_\omega$ [Cohen,Gold], $\mathcal{P}D^{3'}_\omega$ was not investigated but it seems to be easy to show that $\mathcal{P}D^3_\omega = \mathcal{P}D^{3'}_\omega$.) Having a closer look to the reasons for this difference, we observe a fundamental difference in the definition of

ω-languages. For automata the definitions of i-acceptance refer to ω-sequences of states in the finite control. Our definition for Petri nets, however, imposes the corresponding definition on ω-sequences of markings. In automata theoretic terms we did not only restrict the finite control but also the whole memory space.

Therefore we now define the classes $\mathcal{B}_\omega^i$ (and later $\mathcal{K}_\omega^i$) which are defined only by some finite control for the net.

A place $s \in S$ of a net N is *k-bounded* ($k \in \mathbb{N}$), if $m(s) \leq k$ for all reachable markings $m \in (m_0\rangle$. A set of places S_b is *k-bounded* if every place $s \in S_b$ is k-bounded, S_b is *bounded*, if it is k-bounded for some $k \in \mathbb{N}$.

We mention that it is decidable whether a set S_b is bounded nor not [Karp,Miller]. An arbitrary set of bounded places can be seen as the finite control of the net. Therefore we restrict the definition of an i-successful sequence to the markings of a fixed set of places.

If $S_b = \{s_{i_1},\ldots,s_{i_k}\}$ is a subset of $S = \{s_1,\ldots,s_n\}$, then the *projection* $pr_{S_b} : (S \to \mathbb{N}) \to (S_b \to \mathbb{N})$ gives for every marking $m \in M(N)$ the restriction to the places of S_b, i.e. for all $s \in S_b : pr_{S_b}(m)(s) = m(s)$.

Let N be a λ-free Petri net, $S_b \subseteq S$ a bounded set of places and $\mathcal{D} = \{D_1,\ldots,D_k\}$ a finite set of non-empty subsets $D_i \subseteq pr_{S_b}(M(N))$. Then we define for $i \in \{1,1',2,2',3,3'\}$ the *bounded i-behaviour* of $(N,S_b,\mathcal{D})$ by $B_\omega^i(N,S_b,\mathcal{D}) :=$ $\{h(v) \in X^\omega \mid v \in F_\omega(N)$ and $pr_{S_b}(\delta_0(v))$ is i-successful for $\mathcal{D}\}$. The corresponding classes of bounded i-behaviour are denoted by $\mathcal{B}_\omega^i$.

<u>Lemma</u> 4.1: Each class $\mathcal{B}_\omega^i$, for $i \in \{1,1',2,2',3,3'\}$, is closed under finite union.

Often it is easier to state conditions on the sequence of transitions instead of the sequence of markings, because the elements of ω-behaviours of nets are sequences of labels of transitions. Also traditional definitions of fairness base on infinite sequences of transitions. Hence we will introduce an ω-behaviour defined on anchor sets of transitions.

Let N be a λ-free Petri net, $\mathcal{E} = \{E_1,\ldots,E_k\}$ a finite set of non-empty subsets $E_i \subseteq T$, then we define the *transitional i-behaviour* of $(N, \mathcal{E})$ by $K_\omega^i(N, \mathcal{E}) :=$ $\{h(v) \in X^\omega \mid v \in F_\omega(N)$ and v is i-successful for $\mathcal{E}\}$. The corresponding classes of transitional i-behaviour are denoted by $\mathcal{K}_\omega^i$.

<u>Lemma</u> 4.2: Each class $\mathcal{K}_\omega^i$, for $i \in \{1,1',2,2',3,3'\}$, is closed under finite union.

<u>Theorem</u> 4.1: $\mathcal{K}_\omega^i = \mathcal{B}_\omega^i$, for each $i \in \{1,1',1,1',3,3'\}$.

Proof:

1) ($\mathcal{K}^i_\omega \subseteq \mathcal{B}^i_\omega$) Let be $L = K^i_\omega(N, E) \in \mathcal{K}^i_\omega$, with $N = (S,T,F,W,h,m_o)$, $T = \{t_1,\dots,t_n\}$ and $E = \{t_1,\dots,t_k\}$, $k \leq n$.

We will construct a net N' (Figure 4.1) which has the bounded i-behaviour L .

- We introduce k+2 new places called $p_o,p_1,\dots,p_k,p_{k+1}$ which will become the set of bounded (safe) places.
- We build k+2 copies of the set of transitions called $T^o,\dots,T^{k+1}$, let $T^j = \{t^j_1,\dots,t^j_n\}$.
- The transition $t^j_i \in T^j$, $1 \leq i \leq n$, $0 \leq j \leq k+1$, has the same arcs with the places of S as t_i in the original net N , additionally t^j_i removes a token from p_j and fires a token on p_i if $i \leq k$ and on p_{k+1} if $i > k$.
- For the new initial marking there is the marking m_o on the places of S and one token on the place p_o .

It is easy to see that this new net N' has the same behaviour as N . A token on the place p_i , $i \leq i \leq k$, indicates that a copy of $t_i \in E$ was the last fired transition and a token on p_{k+1} that it was a copy of a transition of T/E .

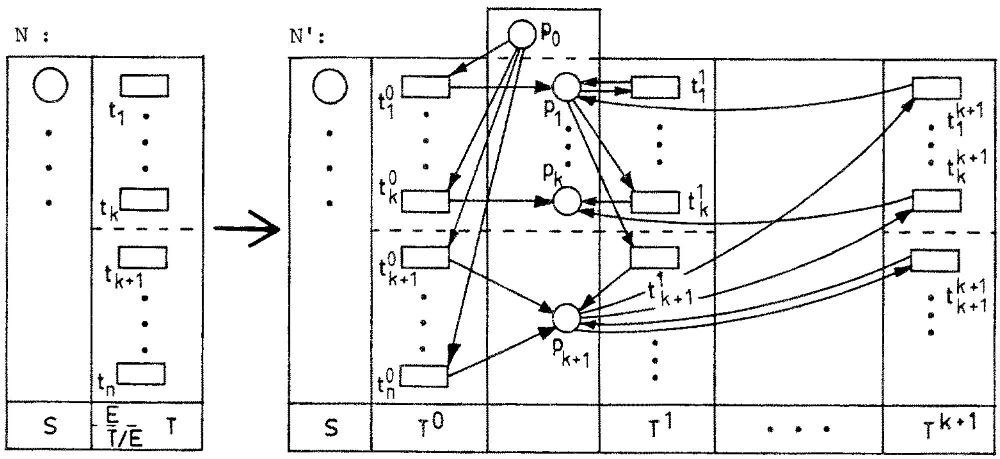


Figure 4.1

Let be $S_b = \{p_0,\dots,p_{k+1}\}$ and m_{pi} , $0 \leq i \leq k$, the marking where only the place p_i contains a token, i.e. $m_{pi}(p_i) = 1$ and $j \neq i$ $m_{pi}(p_j) = 0$.
Let be $D := \{m_{pi} \mid t_i \in E\}$.
Then $L = B^i_\omega(N',S_b,\{D\})$ for $i \in \{1,2,2',3,3'\}$ or $L = B^i_\omega(N'S_b,\{D \cup \{m_{p0}\}\})$ for $i = 1'$.

2) ($\mathcal{B}^i_\omega \subseteq \mathcal{K}^i_\omega$) Let be $L = B^i_\omega(N,S_b,\{D\})$.
We first construct the coverability-graph $G = (V,E,v_o)$ of N as defined in [Jantzen, Valk] (also described in [Valk]). The finite set of vertices V is a

subset of $\mathbb{N}_\omega^{|S|}$ ($\mathbb{N}_\omega := \mathbb{N} \cup \{\omega\}$), the set of edges a subset of $V \times T \times V$.
We build a new net from the coverability-graph G and the original net N in the following way:

- The coverability-graph is transformed into a Petri net (a state machine). The vertices become places, v_o contains a token in the initial marking. Every edge $e = (v_1, t, v_2)$ becomes a transition t_e with $h(t_e) = h(t)$, and arcs (v_1, t_e) and (v_2, t_e).
- This new net will allow more firing sequences than N. Thus the original places of N are added and for every t_e, $e = (v_1, t, v_2)$, the arcs of t in N (it would be sufficient to add only the unbounded places).

Different markings on the bounded places in N have different vertices in G.
For every $d \in D$ let T_d be the set of all transitions which have a vertex v, $pr_{S_b}(v) = d$, as an input place.
Define $\mathcal{E}'$ as the set of all sets E' of transitions so that there is at least one transition of every T_d, $d \in D$, in E'. E' is a subset of the union of all sets T_d, $d \in D$. Then $L = K^1(N', \mathcal{E}')$.

Theorem 4.2 :

a) $\mathcal{L}_\omega = \mathcal{K}_\omega^{1'} \subsetneq \mathcal{K}_\omega^1 = \mathcal{K}_\omega^{2'} \subsetneq \mathcal{K}_\omega^2 = \mathcal{K}_\omega^3 = \mathcal{K}_\omega^{3'}$,

b) There are no more inclusions with the classes $\mathcal{L}_\omega^i$.

Proof: a)

- $\mathcal{L}_\omega \subseteq \mathcal{K}_\omega^{1'}$, let $L = L_\omega(N)$, then $L = K_\omega^{1'}(N, \{T\})$.
- $\mathcal{K}_\omega^{1'} \subseteq \mathcal{L}_\omega$, let $L = K_\omega^{1'}(N, \{E\})$, then L is the ω-behaviour of the net N which has only the transitions of E.
- $\mathcal{L}_\omega \subseteq \mathcal{K}_\omega^1$, let $L = L_\omega(N)$, then $L = K_\omega^1(N, \{T\})$.
- $\mathcal{K}_\omega^1 \not\subseteq \mathcal{L}_\omega$, consider the language $L = a^* b^\omega \in \mathcal{K}_\omega^1$, but $L \notin \mathcal{L}_\omega$.
- $\mathcal{K}_\omega^1 \subseteq \mathcal{K}_\omega^{2'}$, let $L = K_\omega^1(N, \{E\})$, we construct a new net N' according to figure 4.2. Then $L = K_\omega^{2'}(N, \{T2\})$.

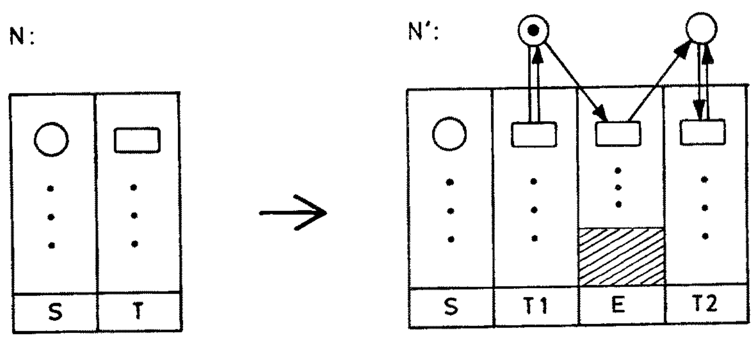


Figure 4.2

- $\mathcal{K}_\omega^{2'} \subseteq \mathcal{K}_\omega^{1}$, let $L = K_\omega^{2'}(N,\{E\})$, we construct a new net N' according to figure 4.3. Then $L = K_\omega^{1}(N,\{T2\})$.

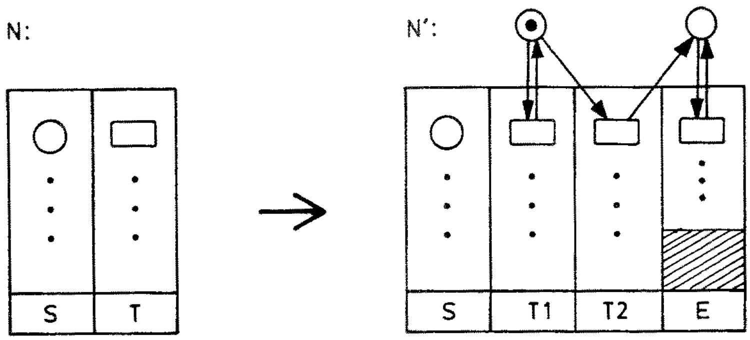


Figure 4.3

- $\mathcal{K}_\omega^{2'} \subseteq \mathcal{K}_\omega^{3}$, let $L = K_\omega^{2'}(N,\{E\})$, we define $\mathcal{E}' := \{E' \mid \emptyset \neq E' \subseteq E\}$, then $L = K_\omega^{3}(N, \mathcal{E}')$.
- $\mathcal{K}_\omega^{2} \not\subseteq \mathcal{K}_\omega^{2'}$, consider the language $L = (a^* b)^\omega \in \mathcal{K}_\omega^{2}$, but $L \notin \mathcal{K}_\omega^{2'}$.
- $\mathcal{K}_\omega^{2} \subseteq \mathcal{K}_\omega^{3'}$, let $L = K_\omega^{2}(N,\{E\})$, we define $\mathcal{E}' := \{\{t\} \mid t \in E\}$, then $L = K_\omega^{3'}(N, \mathcal{E}')$.
- $\mathcal{K}_\omega^{3'} \subseteq \mathcal{K}_\omega^{3}$, let $L = K_\omega^{3'}(N,\{E\})$, we define $\mathcal{E}' := \{E' \mid E \subseteq E' \subseteq T\}$, then $L = K_\omega^{3}(N, \mathcal{E}')$.
- $\mathcal{K}_\omega^{3} \subseteq \mathcal{K}_\omega^{2}$, let $L = K_\omega^{3}(N,\{E\})$, we construct a new net N' according to figure 4.4, then $L = K_\omega^{2}(N,\{T2\})$.

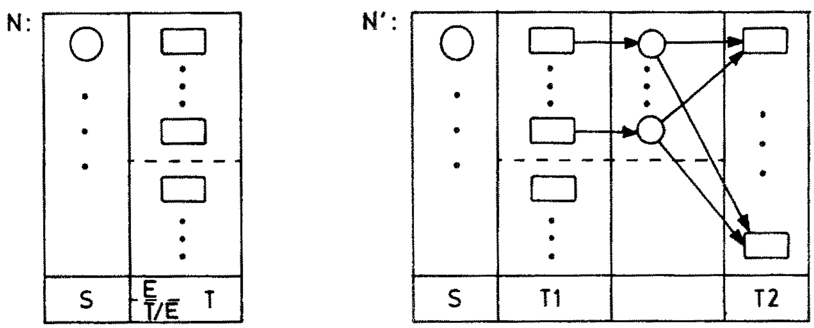


Figure 4.4

b) - $\mathcal{L}_\omega^{2'} \not\subseteq \mathcal{K}_\omega^{2}$, consider the net N in figure 4.5, then $L := L_\omega^{2}(N,\{D\}) = \{a^i b^i a^\omega \mid i \geq 1\}$, and $L \notin \mathcal{K}_\omega^{2}$.

- $\mathcal{K}_\omega^{1} \not\subseteq \mathcal{L}_\omega^{1}$, consider the net N in figure 4.6 with $E := \{t_2\}$, then $L = K_\omega^{1}(N,\{E\}) \notin \mathcal{L}_\omega^{1}$.

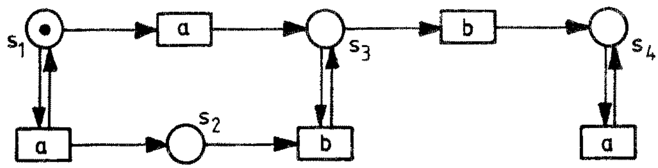


Figure 4.5

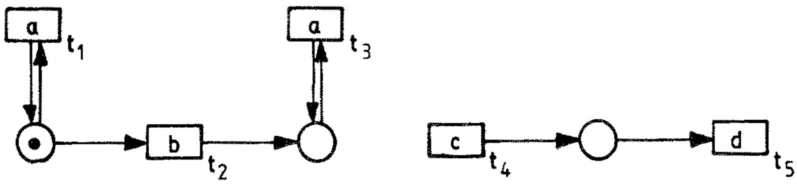


Figure 4.6

To describe fairness by i-behaviour we will look at Dijkstra's well-known problem of the dining five philosophers as an example. A Petri net solution of that problem is shown in figure 4.7. Let N be the net without the dotted part. A fair schedule is obviously a sequence in which every philosopher will eat infinitely often. We may express it by the 3-behaviour of $(N, \mathcal{D})$, where $\mathcal{D} := \{ D \mid \forall j \in \{1,\ldots,5\} \; \exists m \in D: m(eat_j) = 1 \}$. Another problem is shown by the net N', where N' is the net in figure 4.7 including the dotted place 'diff'. This net only allows sequences in which

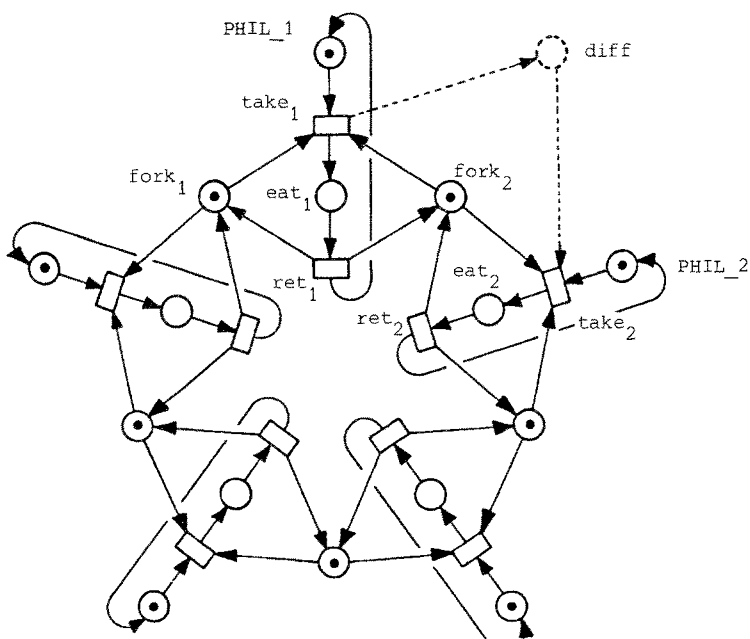


Figure 4.7

'PHIL_1' eats at least as often as 'PHIL_2'. It is not possible to describe the fair behaviour of N' by an i-behaviour. But it is still possible by a bounded 3-behaviour, if we only look at the old places. The same behaviour can be described, however, by a transitional 3'-behaviour. To do this, it is sufficent to consider the singleton anchor set $\mathcal{E} = \{E\}$, where $E = \{take_1, \dots, take_5\}$.

5. A Fair Selection Rule

In the previous sections we introduced some possibilities to describe fair behaviour of a Petri net. We did not investigate how to find a control which forces the net to behave fair.

Now we will look at fairness in the selection of enabled transitions to fire. We will prohibit that enabled transitions are always neglected by introducing a kind of finite delay property for Petri nets. The notations are derived from papers about parallel programming, e.g. [Lehmann,Pnueli,Stavi].

Let N be a λ-free Petri net, then we define an *enabling counter* for an infinite firing sequence and a transition: $en : F_\omega(N) \times T \to \mathbb{N} \cup \{\omega\}$, where $en(v,t) := \left| \{i \in \mathbb{N}^+ \mid \delta_0(v)(i) \geq U(-,t)\} \right|$, and a *continuality predicate:* $con(v,t) := \exists\, i \in \mathbb{N}^+ \; \forall\, j \in \mathbb{N}^+, j \geq i : \delta_0(v)(j) \geq U(-,t)$.

Informally, 'en' gives the number of times a transition is enabled and 'con' is true, if the transition is continuously enabled from some time on.

Now we are able to define languages with fairness in the selection of transitions for the firing rule. We will introduce two new classes of languages.

Let N be a λ-free Petri net, then we define the *just language* of N by $L_\omega^{just}(N) := \{h(v) \in X^\omega \mid v \in F_\omega(N) \text{ and } \forall\, t \in T : con(v,t) \Rightarrow t \in In(v)\}$, and the *fair language* of N by $L_\omega^{fair}(N) := \{h(v) \in X^\omega \mid v \in F_\omega(N) \text{ and } en(v,t) = \omega \Rightarrow t \in In(v)\}$. The corresponding classes of languages are denoted by $\mathcal{L}_\omega^{just}$ and $\mathcal{L}_\omega^{fair}$, respectively.

Note however, that the net N describing the five philosophers' problem in figure 4.7 $L_\omega^{fair}(N)$ is not the fair behaviour as specified in the previous section. In fact there is an infinite firing sequence w where two philosophers, say PHIL_1 and PHIL_3, take their forks in such way that PHIL_2 never has two free forks, i.e. transition $take_2$ is never enabled. Hence w satisfies the definition of the fair language: $w \in L^{fair}(N)$, but PHIL_1 and PHIL_3 behave unfair to PHIL_2. If the net is changed in such a way that the philosophers take their forks one after the other, and avoiding a deadlock by allowing at most four philosophers simultaneously at the table, the fair language of that net will describe a behaviour deserving the name fair.

There are only infinite sequences allowed for $\mathcal{L}_\omega^{just}$ and $\mathcal{L}_\omega^{fair}$. To show the significance of this restriction by an example we mention the net N in figure 5.1.

Figure 5.1

It holds $L_\omega^{just}(N) = \emptyset$ because transition 'b' is always enabled and its firing will lead the net into a deadlock.

Lemma 5.1 : The classes $\mathcal{L}_\omega^{just}$ and $\mathcal{L}_\omega^{fair}$ are each closed under finite union.

We will compare these new classes of Petri net languages with the classes already investigated.

Theorem 5.1 : $\mathcal{K}_\omega^2 \subseteq \mathcal{L}_\omega^{just}$.

Proof: Let $L = K_\omega^2(N,\{E\})$. We construct a new net N' in the same manner as in theorem 4.1 1). Then we add a new place s_{run} as a side condition for every transition, s_{run} contains a token in the initial marking. We also add a new transition which removes a token from s_{run} and p_{k+1}. A firing of this new transition will cause a deadlock. Hence there must be infinitely many situations where it is not enabled, i.e. p_{k+1} does not contain a token. p_{k+1} contains no token iff a transition of E has fired.

Lemma 5.2 : $L1 \in \mathcal{L}_0$ and $L2 \in \mathcal{L}_\omega^{just} \Rightarrow L1 . L2 \in \mathcal{L}_\omega^{just}$.

The proof of this lemma is similar to the proof of theorem 5.1, but in this case we have to ensure that the first net has reached a terminal marking.

Theorem 5.2 : $\mathcal{L}_\omega^{3'} \subseteq \mathcal{L}_\omega^{just}$.

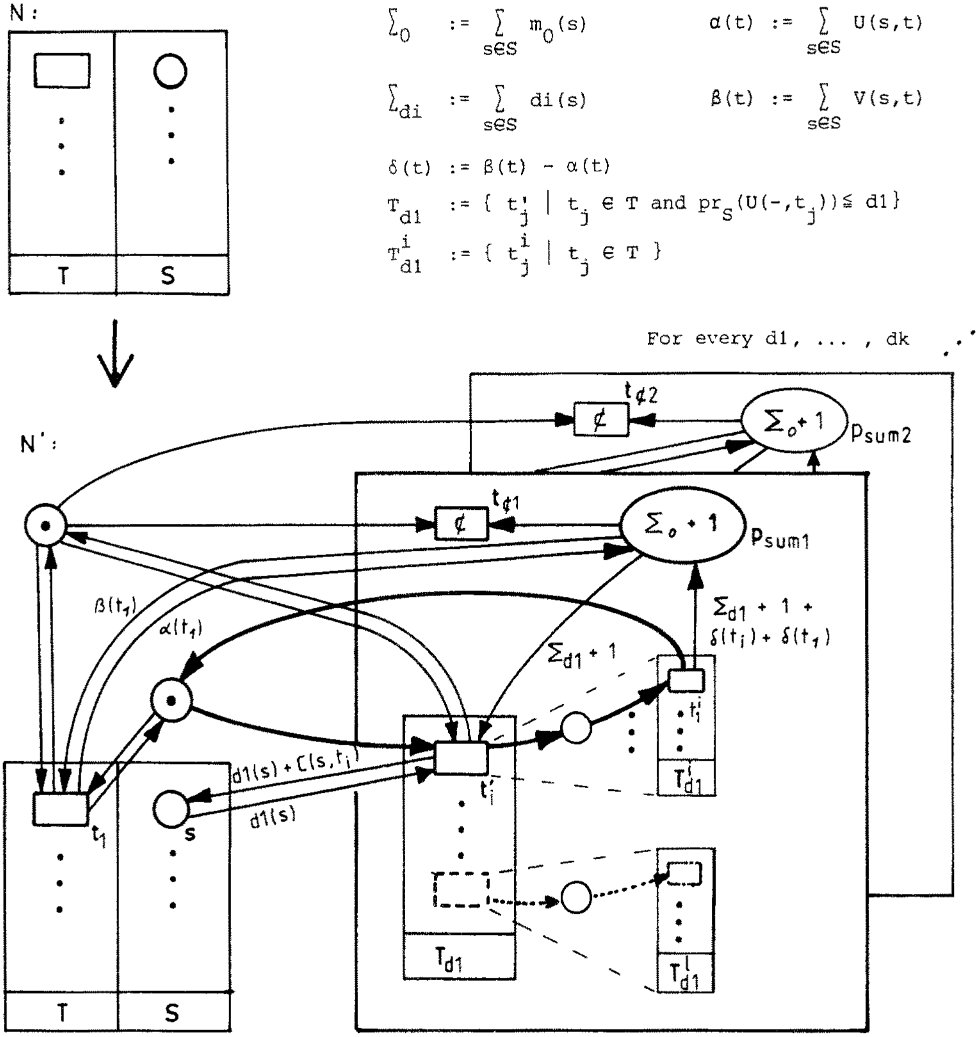
Proof: Let $L = L_\omega^{3'}(N,\{D\})$ with $D = \{d_1,\ldots,d_k\}$ for some $k \geq 1$. We construct a net N' according to the sketch in figure 5.2. We give a short description of the behaviour of the net N':

On the place p_{sumi} there is always one token more than the sum of tokens on the places of S, with the exception that a transition of T_{di} has fired.

For every firing sequence in N' it is possible to find a firing sequence in N with the same word (and vice versa). (Transitions in T_{di} and T_{di}^j may only fire if a corresponding transition in T would be enabled if only the places of S are looked at.)

In N' infinite firing sequences are only possible if no transition $t_{\not{d}i}$ must fire. Hence there must be infinitely often no token on each place p_{sumi}.

Then a transition of T_{di} has fired and there was the marking d_i on the places of S.



$\sum_0 := \sum_{s \in S} m_0(s)$ $\qquad \alpha(t) := \sum_{s \in S} U(s,t)$

$\sum_{di} := \sum_{s \in S} di(s)$ $\qquad \beta(t) := \sum_{s \in S} V(s,t)$

$\delta(t) := \beta(t) - \alpha(t)$

$T_{d1} := \{ t_j' \mid t_j \in T \text{ and } pr_S(U(-,t_j)) \leq d1 \}$

$T_{d1}^i := \{ t_j^i \mid t_j \in T \}$

Figure 5.2

Theorem 5.3 : $\mathcal{L}_\omega^1 \subseteq \mathcal{L}_\omega^{just}$.

Proof: $\mathcal{L}_\omega^1 = \mathcal{L}_0 \circ \mathcal{L}_\omega$ (Theorem 3.1) and $\mathcal{L}_\omega \subseteq \mathcal{L}_\omega^{just}$ (Theorem 5.1 and Theorem 4.2). By lemma 5.2 follows the proposition.

A comparison between the classes of fair and of just behaviours of Petri nets gives the following result.

Theorem 5.4 $\mathcal{L}_\omega^{just} \subsetneq \mathcal{L}_\omega^{fair}$.

To prove the inclusion it is shown that a net N can be transformed into a net N' so that $L_\omega^{just}(N) = L_\omega^{fair}(N')$. The transformation is rather long, it also uses the idea, that every sequence breaking the rules of justice cannot have an infinite fair continuation, i.e. it will bring the net into a deadlock. To show that this inclusion is strict, we take the fair language of the net in figure 5.3.

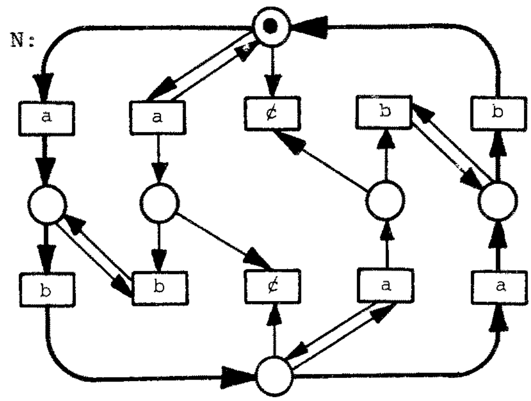


Figure 5.3

Informally we may describe the fair language of this net N that every transition labelled by 'a' or 'b' fires infinitely often where from some time on every sequence of 'a'-transitions is followed by the same number of 'b'-transitions.

$L_\omega^{fair}(N) = \{ w \mid |w|_a = |w|_b \text{ and } \forall i \leq |w| : |w[i]|_a \geq |w[i]|_b \}.$

$(\{a^n b^n a^m b^m \mid n \geq 1 \text{ and } m \geq 1\}^* . \{a^n b^n \mid n \geq 2\})^\omega$

($|w|_a$ denotes the occurences of 'a' in the sequence w) .

There is no Petri net which has such a just behaviour.

In a similar proof one can show that $L = \{a^n b^n \mid n \geq 1\}^\omega \in KC_\omega(\mathcal{L}_0)$, but $L \notin \mathcal{L}_\omega^{fair}$.

Theorem 5.5 : $KC_\omega(\mathcal{L}_0) \not\subseteq \mathcal{L}_\omega^{fair}$.

In the final figure 5.4 we show all inclusions between classes of languages proved in this paper. It was also shown that there are no further inclusions between these classes.

As some concluding remarks we want to mention that for the classes $\mathcal{K}_\omega^i$ [Valk,Jantzen] showed that the emptiness problem is decidable. (That paper also contains additional results on infinite firing sequences, that are continual for some $E \subseteq T$.) For the other classes the emptiness problem is at least as difficult as the reachability problem for Petri nets.

6. Conclusions

In this paper we investigated the inclusions between several classes of ω-behaviour of Petri nets dealing with fairness.

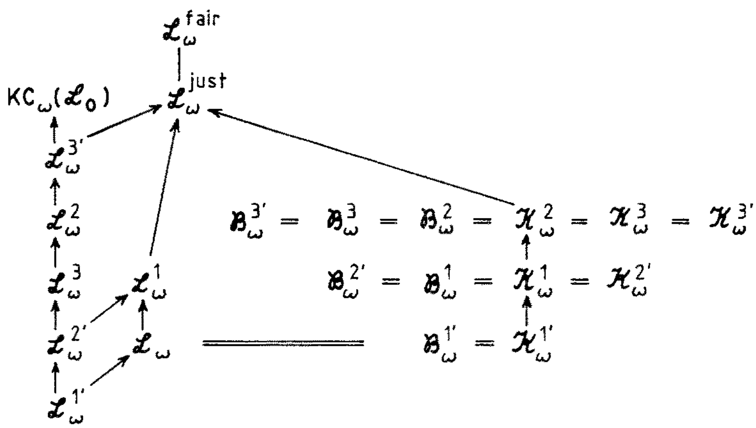


Figure 5.4

An aim for further research will be the relation between the specification of fairness (e.g. markings or transitions which must appear infinitely often) and fairness criteria in the selection of transitions to fire.

An idea is that a specification of fairness can be implemented by nets with some fair selection rule for the transitions.

For an implementation one needs nets which are free of deadlocks. Therefore one question is: what must be changed if we allow only deadlock-free Petri nets? (Certainly a different hierarchy of classes of net languages will follow).

Perhaps it is also more adequate to look at deterministic Petri net languages, as defined in [Vidal-Naquet].

We have investigated two criteria of a fair selection: justice and fairness. Are there other more useful criteria?

For liveness in Petri nets there are many fine results for restriced classes of nets, e.g. for state machine and for free choice nets. Are there also classes of Petri nets in which the problem for fairness becomes much easier as in the general case? Some ideas in that direction are made in [Best] and [Carstensen2]. One such question is : is there a fairness criterion in the selection of transitions for a restricted class of nets which guarantees the infinite firing of certain transitions?

Most of the proofs in this paper are valid for labelled Petri nets only. In these constructions two or more transitions with a same label are used to simulate one transition in the original net. So it would be interesting to see similar results for unlabelled nets.

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