

# A Note on Hack's Conjecture, Parikh Images of Matrix Languages and Multiset Grammars

Georg Zetsche

University of Hamburg, Department of Computer Science  
georg.zetsche@informatik.uni-hamburg.de

## Abstract

It is shown that Hack's Conjecture on Petri nets implies that for every language generated by a matrix grammar (without appearance checking), there is a non-erasing matrix grammar generating a language of the same Parikh image. It is also shown that in this case, the classes of multiset languages generated by arbitrary and monotone multiset grammars coincide.

## Zusammenfassung

Es wird gezeigt, dass die Hack'sche Vermutung über Petri-Netze impliziert, dass es für jede von einer Matrix-Grammatik (ohne Vorkommenstest) erzeugte Sprache eine nicht-löschende solche Grammatik gibt, die eine Sprache vom gleichen Parikh-Bild erzeugt. Es wird auch gezeigt, dass in diesem Fall die Klassen der Multiset-Sprachen, die von beliebigen bzw. monotonen Multiset-Grammatiken erzeugt werden, übereinstimmen.

## 1 Introduction

For definitions and notation, we refer the reader to [Zet09].

In [Hac76, p. 172], Michel Hack conjectured that the reachability problem for Petri nets is decidable. The conjecture also states that this is due to the fact that for every Petri net of size<sup>1</sup>  $y \in \mathbb{N}$ , a constant  $K_y \in \mathbb{N}$  can be determined such that for any marking  $\mu$ , the zero marking is reachable from  $\mu$  iff it can be reached by a firing sequence of length less than  $K_y \cdot \|\mu\|$ .

---

<sup>1</sup>Hack defines the size of a Petri net to be its total number of arcs (see [Hac76, p. 170]).

Although it is known that the reachability problem for P/T nets is decidable (see [May81, May84]), this stronger conjecture remains open and has interesting implications. In this paper, it is shown that if Hack's Conjecture holds, the classes  $\mathbf{MAT}$  and  $\mathbf{MAT}^\lambda$  coincide with respect to their Parikh image. That is, for every matrix grammar  $G$  (without appearance checking), there is a non-erasing matrix grammar  $G'$  such that  $\Psi(L(G')) = \Psi(L(G))$ . In other words,  $\Psi(\mathbf{MAT}^\lambda) \subseteq \Psi(\mathbf{MAT})$  and therefore  $\Psi(\mathbf{MAT}) = \Psi(\mathbf{MAT}^\lambda)$ . Note that whether the classes  $\mathbf{MAT}$  and  $\mathbf{MAT}^\lambda$  are equal is an open question in the theory of regulated rewriting.

The result  $\Psi(\mathbf{MAT}^\lambda) = \Psi(\mathbf{MAT})$  also means that the multiset language classes  $\mathbf{mARB}$  and  $\mathbf{mMON}$  coincide.  $\mathbf{mARB}$  ( $\mathbf{mMON}$ ) is the class of multiset languages generated by arbitrary (monotone) multiset grammars (see [KMVP01] for details on multiset grammars). This result is due to the fact that  $\mathbf{mARB} = \Psi(\mathbf{MAT}^\lambda)$  and  $\mathbf{mMON} = \Psi(\mathbf{MAT})$ .

For more information on Hack's conjecture, see [Gra79].

## 2 Petri net languages

A language of the form  $L_\varphi$  in the next lemma will be needed in one of the later proofs.

**Lemma 1.** *For any monoid-homomorphism  $\varphi : \Sigma^* \rightarrow \mathbb{Z}$ , the language  $L_\varphi = \{w \in \Sigma^* \mid \varphi(w) \geq 0\}$  is in  $\mathcal{L}_0$ .*

*Proof.* We construct a  $\lambda$ -free Petri net  $N = (\Sigma, P, T, \partial_0, \partial_1, \sigma, \mu_0, F)$ . Let  $M := \max\{|\varphi(a)| \mid a \in \Sigma\}$ ,  $\Sigma_- := \{a \in \Sigma \mid \varphi(a) < 0\}$ , and  $\Sigma_+ := \{a \in \Sigma \mid \varphi(a) \geq 0\}$ . The set of transitions is

$$T := \{t_{a,r,s}^+ \mid a \in \Sigma_+, 0 \leq r \leq M, 0 \leq s \leq \varphi(a)\} \\ \cup \{t_{a,r}^- \mid a \in \Sigma_-, 0 \leq r \leq M\}$$

and the set of places is  $P = \{p_+, p_-\}$ . For any  $a \in \Sigma_+$ ,  $0 \leq r \leq M$ ,  $0 \leq s \leq \varphi(a)$ ,  $\sigma(t_{a,r,s}^+) := a$  and for any  $a \in \Sigma_-$ ,  $0 \leq r \leq M$ ,  $\sigma(t_{a,r}^-) := a$ . The initial marking is  $\mu_0 := \mathbf{0}$  and the final markings are

$$F := \{r \cdot (p_+ + p_-) \mid 0 \leq r \leq M\}.$$

The net will work as follows. For  $\mu \in P^\oplus$ , let  $\psi(\mu) := \mu(p_+) - \mu(p_-)$ . If  $\varphi(a) \geq 0$ , the firing of a  $a$ -labeled transition  $t_{a,r,s}^+$  increases the image of the marking under  $\psi$  by a value between 0 and  $\varphi(a)$ . Furthermore, it subtracts  $r$  from both  $p_+$  and  $p_-$ . The latter does not change the image of the marking under  $\psi$  but makes sure that the markings can be kept small.

If  $\varphi(a) < 0$ , then the transitions  $t_{a,r}^-$  add  $\varphi(a)$  to the image of the marking under  $\psi$ . Besides, they subtract a certain value from both places. The pre- and post-multisets are as follows. For  $a \in \Sigma$ ,  $0 \leq r \leq M$ , and, in case  $a \in \Sigma_+$ ,  $0 \leq s \leq \varphi(a)$ , let

$$\begin{aligned} \partial_0(t_{a,r,s}^+) &:= r \cdot (p_- + p_+), & \partial_1(t_{a,r,s}^+) &:= s \cdot p_+, & \text{for } a \in \Sigma_+, \\ \partial_0(t_{a,r}^-) &:= r \cdot (p_- + p_+), & \partial_1(t_{a,r}^-) &:= (-\varphi(a)) \cdot p_-, & \text{for } a \in \Sigma_-. \end{aligned}$$

Let  $\mu \in P^\oplus$  be reachable by a sequence labeled with  $w \in \Sigma^*$ . By induction on the length of  $w \in \Sigma^*$ , one can see that  $\psi(\mu) \leq \varphi(w)$ . The fact that  $\psi(\mu) = 0$  for every  $\mu \in F$  now shows that  $L(N) \subseteq L_\varphi$ .

On the other hand, the following fact is also clear by induction on the length of  $w$ . For every  $w \in \Sigma^*$ , there is a firing sequence  $s$ ,  $\sigma(s) = w$ , that leads to a marking  $\mu$  such that  $\mu(p_+), \mu(p_-) \leq M$  and

- if  $\varphi(w) \geq 0$ , then  $\psi(\mu) = 0$ ,
- if  $\varphi(w) < 0$ , then  $\psi(\mu) = \varphi(w)$ .

Therefore, we have  $L_\varphi \subseteq L(N)$ . □

### 3 Hack's Conjecture

The equality  $\Psi(\mathbf{MAT}) = \Psi(\mathbf{MAT}^\lambda)$  can already be deduced from a slightly weaker version of Hack's Conjecture, which will be stated here.

**Conjecture 2** (Hack's Conjecture). *For every Petri net  $N$ , there is a constant  $K \in \mathbb{N}$  such that for any marking  $\mu$ , the empty marking is reachable from  $\mu$  iff it can be reached by a firing sequence of length less than  $K \cdot \|\mu\|$ .*

The difference between this version and Hack's version is that in the latter, the computability of the constant  $K_y$  from the size  $y$  is also stated. Furthermore, Conjecture 2 only requires the constant  $K$  to depend on  $N$ , whereas Hack's original conjecture states that the constant only depends on the size of  $N$ . This, however, is not a weaker requirement, since, up to initial and final markings, there are only finitely many Petri nets of a certain size. Therefore, if there is such a constant for every Petri net, then there is a constant for every given size.

We will need the following result from the article [Zet09]. It states that applying linear erasing homomorphisms to languages generated by  $\lambda$ -free Petri nets yields languages that can be generated by non-erasing matrix grammars.

**Theorem 3** ([Zet09]).  $\mathcal{H}^{\text{lin}}(\mathcal{L}_0) \subseteq \mathbf{MAT}$ .

The next lemma states that arbitrary matrix languages and Petri net languages have the same Parikh image. For a proof, see [HJ94].

**Lemma 4** ([HJ94]).  $\Psi(\mathbf{MAT}^\lambda) = \Psi(\mathcal{L}_0^\lambda)$ .

The following is the key lemma in our proof, since it describes the consequences of Hack's Conjecture in terms of Petri net languages.

**Lemma 5.** *Suppose Hack's Conjecture holds. Let  $L \subseteq \Sigma^*$  be in  $\mathcal{L}_0$  and  $x \in \Sigma$  be a symbol. Then there is a  $k \in \mathbb{N}$  such that for any word  $w \in L \setminus \{x\}^*$ , there is a  $v \in L$  with  $\Psi(\delta_x(v)) = \Psi(\delta_x(w))$  and  $|v| \leq k \cdot |\delta_x(w)|$ .*

*Proof.* Let  $N = (\Sigma, P, T, \partial_0, \partial_1, \sigma, \mu_0, F)$  be a  $\lambda$ -free Petri net such that  $L(N) = L$  and let  $K$  be the constant from Hack's Conjecture. From  $N$ , we construct a Petri net  $N' = (\Sigma', P', T', \partial'_0, \partial'_1, \sigma', \mu'_0, F')$ , to which we will apply Hack's Conjecture. Let  $\Sigma' := \Sigma \setminus \{x\}$  and let  $p_a$  for every  $a \in \Sigma'$  and  $r$  be new places. Furthermore, let  $t_\mu$  be a new transition for every  $\mu \in F$ , and let  $\mu'_0 := \mu_0 + r$ . The new set of places is then  $P' := P \cup \{p_a \mid a \in \Sigma'\} \cup \{r\}$  and the new set of transitions is  $T' := T \cup \{t_\mu \mid \mu \in F\}$ . For any  $t \in T$  and any  $\mu \in F$ , let

$$\begin{aligned} \partial'_0(t) &:= \begin{cases} r + \partial_0(t) + p_{\sigma(t)} & \text{if } \sigma(t) \neq x, \\ r + \partial_0(t) & \text{otherwise,} \end{cases} & \partial'_1(t) &:= r + \partial_1(t), \\ \partial'_0(t_\mu) &:= r + \mu, & \partial'_1(t_\mu) &:= \mathbf{0}, \\ \sigma'(t) &:= \sigma(t), & \sigma'(t_\mu) &:= \lambda. \end{aligned}$$

The embedding morphism  $\iota : \Sigma'^\oplus \rightarrow P'^\oplus$  is defined by  $\iota(a) := p_a$  for  $a \in \Sigma'$ . Since  $w \notin \{x\}^*$ , we have  $|\delta_x(w)| \geq 1$ . Since  $w \in L$ , there is a firing sequence  $s$  in  $N'$  that leads from  $\nu_w := \mu'_0 + \iota(\Psi(\delta_x(w)))$  to  $\mathbf{0}$ . It follows from the hypothesis that there is also a firing sequence  $s'$  leading from  $\nu_w$  to  $\mathbf{0}$  such that  $|s'| < K \cdot \|\nu_w\|$ . With  $v := \sigma'(s')$ , we have  $\Psi(\delta_x(v)) = \Psi(\delta_x(w))$  and thus

$$\begin{aligned} |v| &\leq |s'| < K \cdot \|\nu_w\| = K \cdot (\|\mu'_0\| + |\delta_x(w)|) = K \cdot (\|\mu'_0\| + |\delta_x(v)|) \\ &\leq K \cdot (\|\mu'_0\| \cdot |\delta_x(v)| + |\delta_x(v)|) = K(\|\mu'_0\| + 1) \cdot |\delta_x(v)|. \end{aligned}$$

Therefore,  $k := K(\|\mu'_0\| + 1) = K(\|\mu_0\| + 2)$  meets our requirements.  $\square$

We will now use the last lemma to infer an inclusion of multiset language classes.

**Lemma 6.** *If Hack's Conjecture holds, then  $\Psi(\mathcal{L}_0^\lambda) \subseteq \Psi(\mathcal{H}^{\text{lin}}(\mathcal{L}_0))$ .*

*Proof.* Let  $L \subseteq \Sigma^*$  be in  $\mathcal{L}_0^\lambda$  and let  $x \in \Sigma$  be a symbol that does not occur in  $L$ . Without loss of generality, we can assume that  $\lambda \notin L$ . Then write  $L = \delta_x(M)$  for some  $M \subseteq \Sigma^*$  in  $\mathcal{L}_0$ . Note that  $M \cap \{x\}^* = \emptyset$ .

For  $M$ , Lemma 5 yields a  $k$  with the property stated there. Let  $R_{\Sigma,k}(x)$  be defined by

$$R_{\Sigma,k}(x) := \{w \in \Sigma^* \mid |w| \leq k \cdot |\delta_x(w)|\},$$

which is in  $\mathcal{L}_0$  according to Lemma 1. Since  $\mathcal{L}_0$  is closed under intersection, the language  $M' = M \cap R_{\Sigma,k}(x)$  is still in  $\mathcal{L}_0$ . The property from Lemma 5 implies  $\Psi(\delta_x(M)) = \Psi(\delta_x(M'))$ . The homomorphism  $\delta_x$  is linear erasing on  $M' \subseteq R_{\Sigma,k}(x)$ . Therefore,  $\Psi(L) = \Psi(\delta_x(M)) = \Psi(\delta_x(M'))$  is in  $\Psi(\mathcal{H}^{\text{lin}}(\mathcal{L}_0))$ .  $\square$

We are now ready to prove the main result.

**Theorem 7.** *If Hack's Conjecture holds, then  $\Psi(\mathbf{MAT}) = \Psi(\mathbf{MAT}^\lambda)$ .*

*Proof.*  $\Psi(\mathbf{MAT}) \subseteq \Psi(\mathbf{MAT}^\lambda)$  follows directly from the definition. Lemma 4, Lemma 6, and Theorem 3 imply

$$\Psi(\mathbf{MAT}^\lambda) = \Psi(\mathcal{L}_0^\lambda) \subseteq \Psi(\mathcal{H}^{\text{lin}}(\mathcal{L}_0)) \subseteq \Psi(\mathbf{MAT}). \quad \square$$

It is a well-known fact that  $\Psi(\mathbf{MAT}^\lambda) = \mathbf{mARB}$  and  $\Psi(\mathbf{MAT}) = \mathbf{mMON}$  (see [KMVP01, Theorem 1]). This yields the following corollary.

**Corollary 8.** *If Hack's Conjecture holds, then  $\mathbf{mARB} = \mathbf{mMON}$ .*

## References

- [Gra79] Jan Grabowski. On Hack's Conjecture Concerning Reachability in Petri Nets. *Elektronische Informationsverarbeitung und Kybernetik*, 15(7):339–354, 1979.
- [Hac76] M. Hack. Decidability Questions for Petri Nets. Technical Report 161, Massachusetts Institute of Technology, Laboratory of Computer Science, June 1976.
- [HJ94] Dirk Hauschildt and Matthias Jantzen. Petri Net Algorithms in the Theory of Matrix Grammars. *Acta Informatica*, (31):719–728, 1994.

- [KMVP01] Manfred Kudlek, Carlos Martín-Vide, and Gheorghe Păun. Toward a Formal Macroset Theory. In *Multiset Processing – Mathematical, Computer Science, and Molecular Computing Points of View*, volume 2235, pages 123–133. 2001.
- [May81] Ernst W. Mayr. An Algorithm for the General Petri Net Reachability Problem. In *STOC '81: Proceedings of the thirteenth annual ACM symposium on Theory of computing*, pages 238–246, New York, NY, USA, 1981. ACM.
- [May84] Ernst W. Mayr. An Algorithm for the General Petri Net Reachability Problem. *SIAM J. Comput.*, 13(3):441–460, 1984.
- [Zet09] Georg Zetsche. Erasing in Petri Net Languages and Matrix Grammars. In Volker Diekert and Dirk Nowotka, editors, *Developments in Language Theory, 13th International Conference, DLT 2009, Stuttgart, Germany, June 30–July 3, 2009. Proceedings*, volume 5583, pages 490–501, 2009.