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A Mereotopological Definition of 'Point'*

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Abstract

Usually, topology is formalised on the basis of set-theoretic notions. But mereology, the formal theory of 'part-of' and related concepts, is a suitable alternative to set theory for this purpose. Thus, formal approaches interrelating mereological and topological notions ('mereotopological approaches') also have a long tradition.

In this paper, a mereotopological definition of 'point' is introduced, based on the topological primitive of 'region'. It is shown that this definition is general enough, such that it allows definitions of all the usual separation properties. In contrast to other proposals in similar frameworks, the relation between points and regions is assumed to be the mereological relation of part-of. In this framework a topological treatment of granularity is possible, in which points at a coarser level of granularity are topologically structured, when analysed on a finer level.

Thus, mereotopology turns out not to be a mere terminological variant of point-set topology but to contribute to the foundations of theories of several domains of interest in cognitive science which exhibit topological structure.

1 Introduction

Motivated by the interest in basically spatial concepts such as 'self-connectedness', 'separation', 'boundary', 'exterior' and 'interior', point-set topology investigates the properties of sets of points, functions between sets of points, their properties and relations etc. This leads to questions such as whether space is a (structured) *set* of points or whether its structure can be adequately represented in terms of sets of points. Since there are at least epistemological reasons to give a negative answer to the former question, one might search for an alternative framework for studying topological concepts. The framework should both allow the definition of

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topological structures without points and not require topological entities to be sets. Classical Mereology, the formal theory of 'part-of' and related concepts, is a sound and suitable basis for this (cf. Tarski 1956, Smith 1993, Varzi 1993, Eschenbach and Heydrich (to appear)). We will call approaches interrelating mereological and topological notions 'mereotopological' and those interrelating mereological and geometrical notions 'mereogeometrical approaches'. The family of such approaches has a long tradition in discussions of the foundations of mathematics (cf. Huntington 1913, de Laguna 1922, Whitehead 1929, Menger 1940, Clarke 1981, 1985).

Although point-set topology is based on the assumption of the existence of points it is not able to contribute to the elucidation of the notion of 'point'. Taking points for granted, it allows for the study of the structure of families of sets of points built up on higher levels. In contrast to this, mereotopological approaches are able to study both the notion of 'point' and topological structures not based on points. Table 1 gives a classification of mereotopological and mereogeometrical approaches with respect to the definition of the notion of 'point'.

Points are Definition is	individuals	classes of individuals	sequences of individuals	classes of sets of individuals
mereological	Euclid Smith 1993			
topological	Eschenbach	de Laguna 1922 Clarke 1985	Menger 1940	Whitehead 1929
geometrical	Huntington 1913	Tarski 1956		

Table 1: Classification of mereotopological definitions of 'point'.

The outline of the paper is as follows: The next two sections will give a brief introduction to Classical Mereology and to the region-based approach of mereotopology. After this, a short discussion concerning the nature of points will prepare the region-based definition of 'point' proposed in section 4. Two sorts of consequence of this definition will be discussed. First, it will be shown that the definition does not restrict the topological structure to topological spaces enjoying certain separation properties only.[1] Second, the discussion of topological structure is related to the discussion of distinguishing different levels of granularity (Hobbs 1985, Habel 1991, 1993) for the representation of space. It will be argued that the approach presented here allows for modelling space in a way, such that an object—on one level of granularity—can be modelled as a point (without any internal topological structure) and—on another level of granularity—as a region with an internal topological structure. Finally, this definition will be contrasted with region-set definitions of 'point' to be found in the literature.

[1]Thus, based on this definition it is possible to differentiate between topological structures without points and topological structures in which points are not separated from each other.

2 Classical Mereology

As argued by Eschenbach and Heydrich (to appear), Classical Mereology (CM) is a formal theory of the relation of 'part-of' and its conceptual relatives 'overlap', 'discreteness' and 'sum', and does not give rise to any concept of integrity or being a whole. CM as introduced by Leśniewski (1916, 1927–30, 1983) is applicable in each and every domain, while concepts of integrity are dependent on the particular features of the structure exhibited by a specific domain. The enrichment of CM by topological notions is an elementary possibility of introducing means for establishing notions of integrity. The concept of self-connectedness, e.g., is a fundamental concept of integrity specifiable on the basis of topological notions. The interrelation of mereological and topological concepts will thus allow for approaching a genuine theory of 'part-whole-relations'.

There have been several approaches to topological or geometrical structures which are not formulated in the framework of point-set topology. Although most of them involve some notion of 'parthood', 'containment' or 'inclusion', most of them are not based on CM. The interrelation of mereological and topological notions can be accomplished in different ways (see Varzi 1993). In several cases, the mereological apparatus needed is gained on the basis of a topological primitive (see de Laguna 1922, Whitehead 1929, Clarke 1981). Consequently, the mereological structure involved in these formalisms need not obey the laws of CM.

Here we will proceed by introducing CM first and using its notion as a basis for defining topological notions (cf. Tarski 1956, Smith 1993, Eschenbach and Heydrich (to appear)). The mereological primitive employed is the binary relation of 'discreteness'. The axiomatisation is close to the one given by Leonard and Goodman (1940).[2]

Primitive notion: *discreteness* ($\wr$)

Definitions:

[D1] x is *part of* y iff x is discrete from everything y is discrete from. ($x < y \equiv_{\text{df}} \forall z\, [z \wr y \supset z \wr x]$)

[D2] x is a *proper part of* y iff x is part of y and y is not part of x. ($x \ll y \equiv_{\text{df}} x < y \wedge \neg(y < x)$)

[D3] x and y *overlap* iff they have a common part. ($x \circ y \equiv_{\text{df}} \exists z\, [z < x \wedge z < y]$)

[D4] x is the *sum* of some entities iff x is discrete from exactly those entities which are discrete from each of them. ($\sigma z\, [\Phi(z)] =_{\text{df}} \iota x\, [\forall y\, [y \wr x \equiv \forall z\, [\Phi(z) \supset y \wr z]]]$, $x + y =_{\text{df}} \sigma z\, [z = x \vee z = y]$)

[D5] x is the *product* of some entities iff x is the sum of all their common parts. ($\pi z\, [\Phi(z)] =_{\text{df}} \sigma y\, [\forall z\, [\Phi(z) \supset y < z]]$, $x \cdot y =_{\text{df}} \sigma z\, [z < x \wedge z < y]$)

[D6] x is an *atom* iff it has no proper part. ($\text{At}(x) \equiv_{\text{df}} \neg \exists y\, [y \ll x]$)

[D7] The *(mereological) complement* of x is the sum of all entities discrete from x. ($x^{-1} =_{\text{df}} \sigma y\, [y \wr x]$)

[2] The logical framework needed for this axiomtisation is that of second order logic with descriptions and identity, including some means for treating non-referring expressions. Plural quantification—as discused by Lewis (1991) with respect to its ontological advantages—is a usefull alternative to quantification over predicates.

[D8] A mereological structure is *atom-free* iff there is no atom.
(Atom-free $\equiv_{\text{df}} \neg\exists x\,[\text{At}(x)]$)

Axioms:

[A1] x and y are discrete iff x and y do not overlap. ($x \wr y \equiv \neg x \circ y$)
[A2] If x is part of y and y is part of x, then x and y are identical.
($x < y \wedge y < x \supset x = y$)
[A3] For any entities, their sum exists. ($\exists x\,[\Phi(x)] \supset \exists y\,[y = \sigma x\,[\Phi(x)]]$)

Since the topic of this paper is not CM, details of the structure defined by [A1–3] are here left aside.[3] For the sake of the discussion of points, four lemmata on properties of atoms should however be mentioned.

Lemma 1: Atom x is part of y iff x overlaps y. ($\forall x, y\,[\text{At}(x) \supset (x < y \equiv x \circ y)]$)

Lemma 2: x is an atom iff x is the product of all entities it overlaps.
($\forall x\,[\text{At}(x) \equiv x = \pi y\,[x \circ y]]$)

Lemma 3: Atoms are equal iff they overlap. ($\forall x, y\,[\text{At}(x) \wedge \text{At}(y) \supset (x = y \equiv x \circ y)]$)

Lemma 4: A mereological structure is atom-free iff every entity overlaps two discrete entites.
(Atom-free $\equiv \forall x\,[\exists y, z\,[x \circ y \wedge x \circ z \wedge y \wr z]]$)

3 Region-based topology

Based on the primitive notion of 'region', fundamental topological notions such as 'boundary', 'open region', 'closed region', 'closure', 'interior', 'exterior', 'separation', 'self-connectedness' etc. can be defined. 'Region' is a unary predicate, not to be understood as implying any specific dimensionality. Thus, it can be true of entities of the highest dimension of the topological structure under consideration. E.g., in studying the topological structure of space, 'region' can be true of three-dimensional entities, and in the case of time, it can be true of one-dimensional ones. Notice that point-set topology has no notion of region.[4] As shall become clear by the following definitions, region-based topology deals with regions and their parts only.[5]

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[3]The structure is a Boolean algebra which is complete with the exception that there is not anything discrete from everything, nothing which is the sum of no entities, and no product of discrete entities, in short: there is no empty individual. Those who are not familiar with CM might, for the time being, imagine the structure defined to be the power-set of some set deprived of the empty set; sum corresponding to union, product to intersection, part-of to subset, overlap to having a non-empty intersection, discreteness to having no common non-empty subset, and atoms to singleton sets. But be aware that CM is more general than this, since it allows for structures without atoms (cf. e.g. Simons 1987), while the interpretation offered here brings atoms with it. Lewis (1991) presents an elaborate discussion of the interrelation between CM and set-theory.

[4]Theorem 19 shows that regions correspond to point-sets which are subsets of the closure of their interior. The union of an open set and a subset of its boundary can therefore be considered a region. Based on the interpretation offered in footnote 3, standard topology can thus be thought of as a specific case of mereotopology.

[5]Parts of regions are called topological entities here. As will be obvious, region-based topology is restricted to that part of the universe which consists of topological entities only. When necessary,

Primitive notion: *region* ($\mathcal{R}$)

Definitions:

[D9] x is a *topological entity* iff x is part of a region. ($\mathcal{T}(x) \equiv_{\text{df}} \exists y\,[\mathcal{R}(y) \wedge x < y]$)

[D10] The *topological universe* is the sum of all regions. ($\mathcal{U}_{\mathcal{R}} =_{\text{df}} \sigma y\,[\mathcal{R}(y)]$)

[D11] Regions x and y are *internally connected* iff they have a common part which is a region. ($x \star y \equiv_{\text{df}} \mathcal{R}(x) \wedge \mathcal{R}(y) \wedge \exists z\,[\mathcal{R}(z) \wedge z < x \wedge z < y]$)

[D12] Regions x and y are *externally connected* iff they overlap but are not internally connected. ($x \times y \equiv_{\text{df}} \mathcal{R}(x) \wedge \mathcal{R}(y) \wedge x \circ y \wedge \neg(x \star y)$)

[D13] Region x is *open* iff it is not externally connected to any region. ($op(x) \equiv_{\text{df}} \mathcal{R}(x) \wedge \neg\exists y\,[x \times y]$)

[D14] The *interior* of x is the sum of the open regions which are part of x. ($x^{\circ} =_{\text{df}} \sigma y\,[op(y) \wedge y < x]$)

[D15] y is *adherent to* x iff every open region which overlaps y overlaps x. ($y \triangleleft x \equiv_{\text{df}} \forall z\,[op(z) \wedge z \circ y \supset z \circ x]$)

[D16] The *closure* of x is the sum of all topological entities adherent to x. ($x^{c} =_{\text{df}} \sigma y\,[\mathcal{T}(y) \wedge y \triangleleft x]$)

[D17] The topological entity x is *closed* iff it is identical to its closure. ($cl(x) \equiv_{\text{df}} (x = x^{c})$)

[D18] The *topological complement* of x is the sum of all topological entities discrete from x. ($x^{-t} =_{\text{df}} \sigma y\,[\mathcal{T}(y) \wedge y \wr x]$)

[D19] The *boundary* of x is the product of its closure and the closure of its topological complement. ($x^{b} =_{\text{df}} x^{c} \cdot (x^{-t})^{c}$)

[D20] The topological entities x and y are *separated* iff x is discrete from the closure of y and y is discrete from the closure of x. ($x \parallel y \equiv_{\text{df}} \mathcal{T}(x) \wedge \mathcal{T}(y) \wedge x^{c} \wr y \wedge x \wr y^{c}$)

[D21] The topological entity x is *self-connected* iff it is not the sum of two separated topological entities. ($con(x) \equiv_{\text{df}} \mathcal{T}(x) \wedge \neg\exists z, y\,[z + y = x \wedge z \parallel y]$)

[D22] y is an *inner part* of the topological entity x iff y is part of an open region z which is part of x. ($y <_{i} x \equiv_{\text{df}} \mathcal{T}(x) \wedge \exists z\,[op(z) \wedge y < z \wedge z < x]$)

[D23] If x is a topological entity, then y is a *dangling part* of x iff y is a proper part of x, x has a topological complement z, and y is an inner part of the closure of z. ($y <_{dg} x \equiv_{\text{df}} \mathcal{T}(x) \wedge y \ll x \wedge \exists z\,[z = x^{-t} \wedge y <_{i} z^{c}]$)

[D24] A topological space is *grounded on closed entities* iff every region is the sum of the closed entities which are part of it. ($\text{GoCE} \equiv_{\text{df}} \forall x\,[\mathcal{R}(x) \supset x = \sigma y\,[cl(y) \wedge y < x]]$)

Different restrictions on the primitive of 'region' lead to different topologies just as different restrictions on 'open set', 'neighbourhood' or 'closure' in classical topology do. The axioms [A4–6] seem to be essential if the structure is to be called 'topological' at all.

Axioms:

[A4] Every region has an open region as a part. ($\forall x\,[\mathcal{R}(x) \supset \exists y\,[op(y) \wedge y < x]]$)

quantification is restricted explicitly to allow for an easy embedding of topology into an more comprehensive ontological framework.

[A5] For any regions their sum is a region. ($\forall x\,[x = \sigma y\,[\mathcal{R}(y) \wedge y < x] \supset \mathcal{R}(x)]$)
[A6] The product of any two overlapping open regions is an open region. ($\forall x, y\,[op(x) \wedge op(y) \wedge x \circ y \supset op(x \cdot y)]$)

Example 1: Let a, b, c be three mutually discrete entities and $a, b, a+b, a+c, b+c, a+b+c$ the regions. According to the definitions given above, $a+b+c$ is the topological universe and any part of it is a topological entity. $a+c$ and $b+c$ are externally connected, while $a+c$ and $a+b$ are internally connected. $a, b, a+b, a+b+c$ are open regions, and $c, a+c, b+c, a+b+c$ are closed entities. $a, b, a+b+c$ are the interiors of $a+c, b+c$ and $a+b+c$, respectively. c is adherent to $a, b, c, a+b, a+c, b+c, a+b+c$, but has nothing but itself adherent to it. $a+c, b+c, a+b+c$ are the closures of a, b and $a+b$, respectively. c is the boundary of $a, b, c, a+c, b+c$ and $a+b$, but not of $a+b+c$. $a+b$ is not self-connected, since a and b are separated. But neither a and $b+c$, nor b and $a+c$, nor c and $a+b$ are separated. Therefore $a+b+c$ is self-connected.

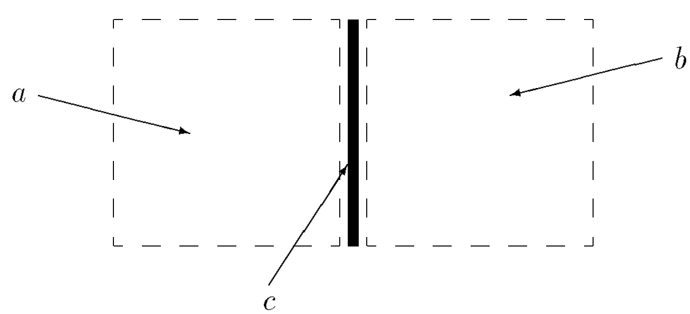

Figure 1: Example 1.

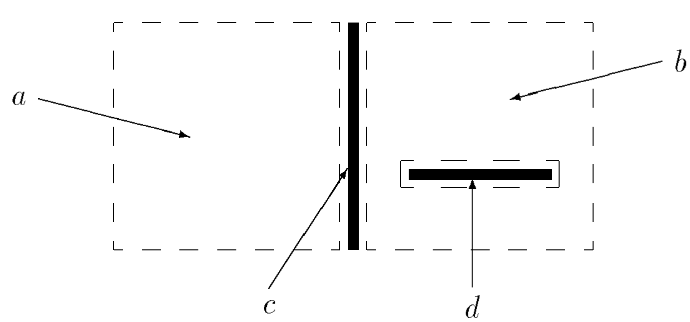

Figure 2: Example 2.

Example 2: Let a, b, c, d be four mutually discrete entities and $a, b, a+b, a+c, b+c, b+$

$d, a+b+c, b+c+d, a+b+c+d$ be the regions. $a+b+c+d$ is the topological universe and any part of it is a topological entity. $a, b, a+b, b+d, a+b+d, a+b+c, a+b+c+d$ are open regions and $c, d, a+c, b+c+d, a+b+c+d$ are closed entities. $a, b, b+d, a+b+c$ are the interiors of $a+c, b+c, b+d+c$ and $a+b+c$, respectively. c is adherent to $a, b, c, a+b, a+c, b+c, a+b+c$, but has nothing but itself adherent to it. d is, e.g., adherent to $b, d, a+b, b+c, a+b+c$, but has nothing but itself adherent to it. $a+c, b+c+d, a+b+c+d$ are the closures of a, b and $a+b$, respectively. c is the boundary of $a, b+d, c, a+c, b+c+d$ and $a+b+d$. d is the boundary of d and $a+b+c$. $c+d$ is the boundary of $b, a+b$ and $b+c$. a and b are separated and, thus, $a+b$ is not self-connected. But neither a and $b+c$, nor b and $a+c$, nor c and $a+b$ are separated. Therefore $a+b+c$ is self-connected. b and d are inner parts of $b+d$ and d is a dangling part of $a+d$.

The structure defined by these axioms is in many respects similar to classical point-set topology, although, as standardly in approaches based on Classical Mereology, there is nothing like an empty topological entity (the mereological structure does not include an empty object). The mereotopological counterparts of well known topological properties and interrelations are otherwise valid.

Theorem 5: The topological universe is open and closed. $(op(\mathcal{U}_{\mathcal{R}}) \wedge cl(\mathcal{U}_{\mathcal{R}}))$

Theorem 6: The sum of open regions is open. $(\exists x\,[\Phi(x)] \wedge \forall x\,[\Phi(x) \supset op(x)] \supset op(\sigma x\,[\Phi(x)]))$

Theorem 7: x is an open region iff x is identical to its interior. $(\forall x\,[op(x) \equiv x = x^{\circ}])$

Theorem 8: The sum of two closed entities is closed. $(\forall x, y\,[cl(x) \wedge cl(y) \supset cl(x+y)])$

Theorem 9: If x is the product of closed entities, then x is closed.
$(\exists y\,[\Phi(y)] \wedge \forall y\,[\Phi(y) \supset cl(y)] \supset \forall x\,[x = \pi y\,[\Phi(y)] \supset cl(x)])$

Theorem 10: The closure of x is the product of all closed topological entities x is part of. $(\forall x\,[x^{c} = \pi y\,[cl(y) \wedge x < y]])$

Theorem 11: The interior of x is part of x. $(\forall x, y\,[y = x^{\circ} \supset y < x])$

Theorem 12: If x is part of the topological entity y, then x is adherent to y. $(\forall x, y\,[\mathcal{T}(y) \wedge x < y \supset x \lhd y])$

Theorem 13: If x is a topological entity, then x is part of the closure of x.
$(\forall x\,[\mathcal{T}(x) \supset x < x^{c}])$

Theorem 14: If x is a topological entity and y the topological complement of x, then x is open iff y is closed, and x is closed iff y is open.
$(\forall x, y\,[\mathcal{T}(x) \wedge y = x^{-t} \supset (op(x) \equiv cl(y)) \wedge (cl(x) \equiv op(y))])$

Theorem 15: If the topological space is grounded on closed entities, then y is adherent to x iff no part of y is separated from x.
$(\mathrm{GoCE} \supset \forall x, y\,[y \lhd x \equiv \neg\exists z\,[z < y \wedge z \parallel x]])$

Notice that regions need not be self-connected. The following theorems are meant to shed some light on what can be conceived of as a region. In essence, a region is an entity whose boundary is (entirely) adherent to its interior and is not too thick (i.e. has no region as a part).

Lemma 16: If there is a region, then the topological universe is a region. $(\exists x\,[\mathcal{R}(x)] \supset \mathcal{R}(\mathcal{U}_{\mathcal{R}}))$

Lemma 17: x is a topological entity iff x is part of the topological universe. $(\forall x\,[\mathcal{T}(x) \equiv x < \mathcal{U}_{\mathcal{R}}])$

Lemma 18: Every region is a topological entity. $(\forall x\,[\mathcal{R}(x) \supset \mathcal{T}(x)])$

Theorem 19: If x is a region, then it has no dangling parts. $(\forall x\,[\mathcal{R}(x) \supset \neg\exists y\,[y <_{dg} x]])$

Theorem 20: If x is a region and y the boundary of x, then y has no region as a part. $(\forall x, y\,[\mathcal{R}(x) \wedge y = x^b \supset \neg\exists z\,[\mathcal{R}(z) \wedge z < y]])$

4 Points

Euclid's definition (a point is that which has no (proper) part) is what first comes to mind when we are asked what is a point.[6] This definition is a purely mereological one, and there already is a place for this kind of entity in the above: they are atoms. But although 'point' should be defined in a way that is based on topological notions, it is worth comparing atoms and their properties to points.

Atom are those entities which do not exhibit any structure at all: They do not have any proper part and do not overlap another atom nor any entity they are not a part of. Points are entities which do not exhibit any *topological* structure: They do not have a proper part which is a region or a point, they do not overlap another point nor overlap any region they are not in.

An alternative definition of 'point' is a (geo)metrical one: A point is what is not extended. This definition is based on a metrical—and not a purely topological—concept. Since geometrical and metrical structures are richer than topological ones in the sense that topological concepts can be defined in (geo)metrical terms, it is worth looking for a basically topological definition.[7]

A definition of 'point' is appropriate if it is in accordance with the intuitive notion we have. An intuitively clear mereotopological definition of 'point' is one which takes points to be regions which have no region as proper part. Such regions do not exhibit any topological structure. A corresponding idea is expressed by Huntington in the context of mereogeometry.[8] But there are topological structures which do not involve such regions, and therefore it seems worth search for a more general definition.

The relation between points and regions is often specified by the preposition *in*. This relation can be understood as being the mereological relation of part-of, which makes points topological entities. In addition, it seems to be sensible to

[6]Cf. Smith (1993), a recent approach in mereotopology along this line.

[7]Although at this stage of discussion it is not possible to show that the definition presented here fulfils this requirement, it should be noted that an appropriate definition of 'point' should lead to the validity of: If x is a point with respect to the topological structure arisen by a metrical space (or induced by the metric), then x is not extended with respect to the underlying metric.

[8]Taking *sphere* and *containment* as primitive notions, Huntington (1913) defines points to be spheres which do not contain a(nother) sphere.

assume that every point is in at least one region. This makes points ontologically dependent on regions. An identity criterion for points can also be based on region: Points which are in exactly the same regions are equal. As said before, although points are topological entities, they do not exhibit any topological structure. These requirements are met by definition [D25].

Definitions:

[D25] x is a *point* iff x is a topological entity, x is part of every region it overlaps, and every topological entity which is part of any region x overlaps is part of x.
$(PT(x) \equiv_{\mathrm{df}} \mathcal{T}(x) \wedge \forall z\,[\mathcal{T}(z) \supset (z < x \equiv \forall y\,[\mathcal{R}(y) \wedge x \circ y \supset z < y])])$

[D26] x is *in* y iff x is a point, y a region, and x is part of y.
$(IN(x,y) \equiv_{\mathrm{df}} PT(x) \wedge \mathcal{R}(y) \wedge x < y)$

[D27] z is a *neighbourhood* of point x iff z is a topological entity and x is in an open region y which is part of z.
$(\mathcal{N}(z,x) \equiv_{\mathrm{df}} PT(x) \wedge \mathcal{T}(z) \wedge \exists y\,[op(y) \wedge IN(x,y) \wedge y < z])$

[D28] A topological space is *point-free* iff there is no point in the topological universe. $(\mathrm{Pfr} \equiv_{\mathrm{df}} \neg\exists x\,[PT(x) \wedge IN(x,\mathcal{U}_{\mathcal{R}})])$

[D29] A topological space is *grounded on points* iff every region is the sum of the points in it. $(\mathrm{GoP} \equiv_{\mathrm{df}} \forall y\,[\mathcal{R}(y) \supset y = \sigma x\,[PT(x) \wedge IN(x,y)]])$

The topological structure defined by [A4–6] is not restricted in any way with respect to the existence of points. The structure may allow for topological differentiation without limit. Point-free topology is possible[9], as well as topology which includes points but is not grounded on them.

That points can have parts is shown in the following example.

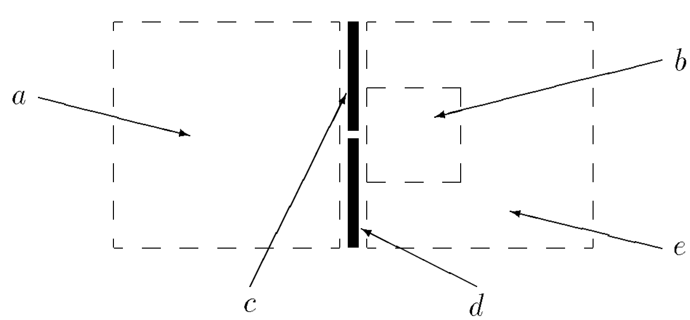

Figure 3: Example 3.

Example 3: Let a, b, c, d, e be five mutually discrete entities, $a, b+e, a+b+e, a+c+d, b+c+d+e, a+b+c+d+e$ the regions. $a, b+e, a+b+e, a+b+c+d+e$ are open

[9] A simple case of point-free topology is an atom-free mereological structure in which every entity is a region.

regions and $a+c+d, b+c+d+e, a+b+c+d+e$ the closures of $a, b+e, a+b+e$ respectively. $c+d$ is the closure of c, d and itself and $c+d$ is the boundary of $a, b+e, a+c+d, b+c+d+e, a+b+e$. In this structure, there are three points: $a, b+e, c+d$, and b, c, d, e are proper parts of points.

Theorems 21 and 22 show that Huntington's (1913) definition of 'point' captures a special case of the definition presented here.

Theorem 21: If x is a region which has no region as proper part, then x is a point. $(\forall x\,[\mathcal{R}(x) \wedge \neg\exists y\,[\mathcal{R}(y) \wedge y \ll x] \supset PT(x)])$

Theorem 22: If x is a region which does not overlap any region it is not part of, then x is a point. $(\forall x\,[\mathcal{R}(x) \wedge \neg\exists y\,[\mathcal{R}(y) \wedge x \circ y \wedge \neg x < y] \supset PT(x)])$

The lack of topological structure of points is expressed in theorems 23, 24, 28 and 29. The following theorems show the similarity of points and atoms and that the definition meets the analysis of the intuitive notion of 'point'.

Theorem 23: Point x is in region y iff x overlaps y.
$(\forall x, y\,[PT(x) \wedge \mathcal{R}(y) \supset (IN(x, y) \equiv x \circ y)])$

Theorem 24: The topological entity x is a point iff x is the product of all regions it overlaps. $(\forall x\,[PT(x) \equiv (x = \pi y\,[\mathcal{R}(y) \wedge x \circ y])])$

Theorem 25: Points are equal iff they overlap.
$(\forall x, y\,[PT(x) \wedge PT(y) \supset (x = y \equiv x \circ y)])$

Theorem 26: A topological space is point-free iff every topological entity overlaps two discrete regions. $(\mathrm{Pfr} \equiv \forall x\,[\mathcal{T}(x) \supset \exists y, z\,[x \circ y \wedge x \circ z \wedge y \wr z]])$

Theorem 27: Points are equal iff they are in exactly the same regions.
$(\forall x, y\,[PT(x) \wedge PT(y) \supset (x = y \equiv \forall z\,[\mathcal{R}(z) \supset (IN(x, z) \equiv IN(y, z))])])$

Theorem 28: If x is a point and y part of x, then x is part of the closure of y. $(\forall x, y\,[PT(x) \wedge y < x \supset x < y^{c}])$

Theorem 29: Every point is self-connected. $(\forall x\,[PT(x) \supset con(x)])$

That no atoms are part of the topological universe does not mean that there are no points in it. Just take any example given and assume that the basic, non-analysed objects are infinitely dividable. On the other hand, that there are no points in the topological universe means that there are no atoms which are part of it. The validity of theorem 31 poses a serious restriction on the possibility of presenting examples of point-free topological structures. First, topologies derived from finitely many entities (atoms) are grounded on points. Second, if you think of the domain of Classical Mereology as sketched in footnote 3, standard topology can be thought of as mereotopology. But since singleton sets are atoms with respect to this structure, in this domain one will always get atomic structures and, accordingly, topological structures based on points.

Theorem 30: If the topological entity x is part of every region it overlaps, then x is part of a point. $(\forall x\,[\mathcal{T}(x) \wedge \forall z\,[\mathcal{R}(z) \wedge x \circ z \supset x < z] \supset \exists y\,[PT(y) \wedge x < y]])$

Theorem 31: Every atom which is a topological entity is part of a point.
$(\forall x\,[\mathcal{T}(x) \wedge \mathrm{At}(x) \supset \exists y\,[PT(y) \wedge x < y]])$

Theorems 32 to 35 show that the relation between points and their neighbourhoods corresponds to the classical topological one.

Theorem 32: A point is inner part of every one of its neighbourhoods.
$(\forall x, y\, [PT(x) \wedge \mathcal{N}(y,x) \supset x <_i y])$

Theorem 33: Every topological entity which has a neighbourhood of point x as a part is a neighbourhood of x. $(\forall x, y\, [PT(x) \wedge \exists z\, [\mathcal{N}(z,x) \wedge z < y] \supset \mathcal{N}(y,x)])$

Theorem 34: The product of two neighbourhoods of a point x exists and is a neighbourhood of x. $(\forall x, y, z\, [PT(x) \wedge \mathcal{N}(y,x) \wedge \mathcal{N}(z,x) \supset \exists w\, [w = y \cdot z \wedge \mathcal{N}(w,x)]])$

Theorem 35: Every neighbourhood z of a point x has a neighbourhood y of x as a part, such that z is neighbourhood of every point which is part of y.
$(\forall x, z\, [PT(x) \wedge \mathcal{N}(z,x) \supset \exists y\, [\mathcal{N}(y,x) \wedge y < z \wedge \forall w\, [PT(w) \wedge w < y \supset \mathcal{N}(z,w)]]])$

In the approach presented, the relation between points and regions or boundaries is the mereological relation of part-of. But the definition of 'point' does not imply that every point is, e.g., a part of at least one boundary. In addition, the question whether every region or boundary is exclusively built up from points (i.e. is the sum of the points which are part of it) might be discussed. Therefore, it is possible to study whether a topological structure allows for something other than points (i.e. for something not having any point as part or, even stronger, something not overlapping any point).

5 Points and Separation

Theorem 27 suggests that points are always separated in a sense analogous to the condition on T_0-spaces in point-set topology.[10] From the definition of 'point' it might be suspected that points are even separated according to the demands of T_1-spaces.[11] To show that this is not the case, first the definitions of the separation properties have to be transferred to region-based topology. It is important to notice that the mereotopological analogue of the condition on normal spaces in region-based topology is independent from what points are and whether any exist.

Definitions:

[D30] A topological space is a T_0-*space* iff for every two (distinct) points there is an open region z such that exactly one of them is in z.
$(T_0 \equiv_{\mathrm{df}} \forall x, y\, [PT(x) \wedge PT(y) \wedge x \neq y \supset \exists z\, [op(z) \wedge (IN(x,z) \not\equiv IN(y,z))]])$

[D31] A topological space is a T_1-*space* iff for every two (distinct) points x, y there is an open region z such that x is in z and y is not in z.
$(T_1 \equiv_{\mathrm{df}} \forall x, y\, [PT(x) \wedge PT(y) \wedge x \neq y \supset \exists z\, [op(z) \wedge IN(x,z) \wedge \neg IN(y,z)]])$

[10] A point-set topological space is a T_0-space iff for every two (distinct) points there is an open set z such that exactly one of them is in z.

[11] A point set topological space is a T_1-space iff for every two (distinct) points x, y there is an open set z such that x is in z and y is not in z.

[D32] A topological space is a T_2- (or *Hausdorff-)space* iff for every two (distinct) points x, y there are two discrete open regions u, z such that x is in z and y is in u.
($T_2 \equiv_{\mathrm{df}} \forall x, y\,[PT(x) \wedge PT(y) \wedge x \neq y \supset$
$\exists u, z\,[op(z) \wedge op(u) \wedge u \wr z \wedge IN(x, z) \wedge IN(y, u)]]$)

[D33] A topological space is *regular* iff for every closed entity y and point x not in y there are two discrete open regions u, z such that x is in z and y is part of u.
(Regular $\equiv_{\mathrm{df}} \forall x, y\,[PT(x) \wedge cl(y) \wedge \neg IN(x, y) \supset$
$\exists u, z\,[op(z) \wedge op(u) \wedge u \wr z \wedge IN(x, z) \wedge y < u]]$)

[D34] A topological space is a T_3-*space* iff it is a T_1-space and regular.
($T_3 \equiv_{\mathrm{df}} T_0 \wedge$ Regular)

[D35] A topological space is *normal* iff for any two discrete closed entities y, x there are two discrete open regions u, z such that x is part of z and y is part of u.
(Normal $\equiv_{\mathrm{df}} \forall x, y\,[cl(x) \wedge cl(y) \wedge x \wr y \supset$
$\exists u, z\,[op(z) \wedge op(u) \wedge u \wr z \wedge x < z \wedge y < u]]$)

[D36] A topological space is a T_4-*space* iff it is a T_1-space and normal.
($T_4 \equiv_{\mathrm{df}} T_0 \wedge$ Normal)

Points need not be separated since they might be in exactly the same *open* regions while there is a non-open region only one of them is in. Whether a topological structure is a T_0- or T_1-space depends on whether or not we add the condition that points are closed entities.

Theorem 36: There are mereotopological spaces which are not T_0-spaces.

Proof of 36: Let a, b, c, d be four mutually discrete entities, $a, b, a+b, a+c, a+d, a+c+d, b+c+d, a+b+c+d$ the regions. In this structure, there are four points: a, b, c, d. $a+b+c+d$ is the only open region c or d are in. Thus, c and d are in exactly the same open regions and therefore not separated. Q.E.D.

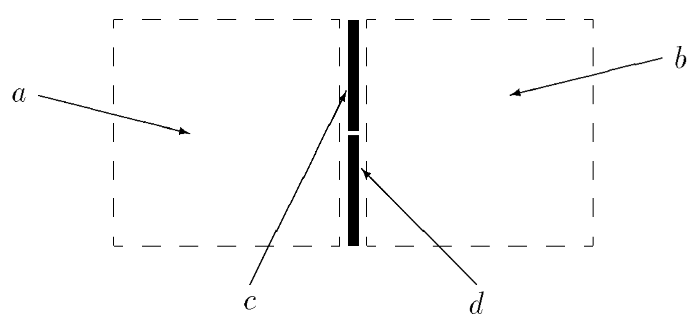

Figure 4: Proof of theorem 36.[12]

[12]The figure might be misleading, since it suggests that the relation between a and c and d is

Theorem 37: If every point of a topological space is closed, then it is a T_1-space. $(\forall x\,[PT(x) \supset cl(x)] \supset T_1)$

Theorem 38: If a T_1-space is grounded on points, then every point is closed. $(T_1 \wedge \mathrm{GoP} \supset \forall x\,[PT(x) \supset cl(x)])$

Theorem 39: If every point of a topological space is closed, then the topological space is a T_3-space iff it is regular. $(\forall x\,[PT(x) \supset cl(x)] \supset (T_3 \equiv \mathrm{Regular}))$

Theorem 40: If every point of a topological space is closed, then the topological space is a T_4-space iff it is normal. $(\forall x\,[PT(x) \supset cl(x)] \supset (T_4 \equiv \mathrm{Normal}))$

6 Topological levels

In several areas of cognitive science it has been noted that there is a need to take different levels of granularity or refinement into account (Hobbs 1985, Habel 1991, 1993). Knowledge and belief about several domains can be organised according to which level of detail is needed for expressing it. To give an example from the area of knowledge about space: Maps of countries present (smaller) cities as points (without internal structure), while maps of these cities show a lot of structure, and, in turn, present houses, blocks and (smaller) parks as unstructured. The question addressed in this paragraph is, how mereotopological structures on different levels of granularity can be interrelated. The introduction of different levels of granularity should allow for topological entities which are points on the coarser level, while exhibiting topological structure on the finer level.

The comparison of different topological structures of one domain can also be done in the framework of point-set topology.[13] Although there might be immense differences between different structures of one domain, all structures have the same notion of point. This holds because what is a point in classical topology is only dependent on what the domain is. In contrast to this, mereotopology allows for topological entities which are points on the coarser level and structured regions on a finer level of granularity.

The mereological structure assumed as basis for defining the topological structure is domain independent and also independent of levels of granularity. If an entity has parts, than it has these parts, independently of whether they are of interest on the level of granularity under consideration or not. Different levels of granularity in topologically structured domains must therefore be differentiated on the basis of the topological notions. In the region-based approach to mereotopology proposed here, one may get different sets of points by introducing different notions of 'region'. (In what follows, the subscripts $_c$ and $_f$ are used to refer to the topological structures imposed by $\mathcal{R}_c$ and $\mathcal{R}_f$.)

There are different ways in which systems of regions can be interrelated. The interrelation between different topological levels in one system of granularity should

the same as that between b and c and d. That this is not the case should be clear from the explicit definition.

[13] A topological structure is called 'coarser' than another, if every set which is open wrt. the former is open wrt. the latter structure.

obey axiom [A7], which forces the topological universe of the different levels of granularity to be invariant, and axiom [A8], which motivates the terms 'finer' and 'coarser' level of granularity.

Axioms:

[A7] The topological universe is the same on all levels of granularity. ($\mathcal{U}_{\mathcal{R}_f} = \mathcal{U}_{\mathcal{R}_c}$, $\sigma y\,[\mathcal{R}_f(y)] = \sigma y\,[\mathcal{R}_c(y)]$)

[A8] Any region at the coarser level is a region at the finer level. ($\forall x\,[\mathcal{R}_c(x) \supset \mathcal{R}_f(x)]$)

As a consequence of these axioms, the following theorems are provable.

Theorem 41: Any topological entity on the finer level is a topological entity on the coarser level and vice versa. ($\forall x\,[\mathcal{T}_f(x) \equiv \mathcal{T}_c(x)]$)

Theorem 42: If x is the topological complement of y on one level, then x is the topological complement of y on the other level of granularity. ($\forall x, y\,[x = y^{-t_f} \equiv x = y^{-t_c}]$)

Theorem 43: If x and y are internally connected on the coarser level, then they are internally connected on the finer level. ($\forall x, y\,[x \star_c y \supset x \star_f y]$)

Theorem 44: If x is a point on the finer level, then x is part of a point at the coarser level. ($\forall x\,[PT_f(x) \supset \exists y\,[PT_c(y) \wedge x < y]]$)

The topological granularity system allows for many different kinds of structures. It is, e.g., not provable, that boundaries on the finer level are less (or 'thinner') than those on the coarser level. It is also possible that an open region on the coarser level is not open on the finer level. Thus, the relation between topological structures as defined by [A7] and [A8] differs from the relation 'finer–coarser' as standardly defined on the basis of classical topology. The relation corresponding to the classical one is gained by the addition of one of the following statements, which are equivalent to each other in the given context, as an axiom.

- Every open region on the coarser level is open on the finer level. ($\forall x\,[op_c(x) \supset op_f(x)]$)
- Every closed entity on the coarser level is closed on the finer level. ($\forall x\,[cl_c(x) \supset cl_f(x)]$)
- If y is the interior of x at the coarser level, then y is part of the interior of x at the finer level. ($\forall x, y, z\,[y = x^{\circ_c} \wedge z = x^{\circ_f} \supset y < z]$)
- If y is adherent to x at the finer level, then y is adherent to x at the coarser level. ($\forall x, y\,[y \lhd_f x \supset y \lhd_c x]$)
- If y is the closure of x at the finer level, then y is part of the closure of x at the coarser level. ($\forall x, y, z\,[y = x^{c_f} \wedge z = x^{c_c} \supset y < z]$)
- If y is the boundary of x at the finer level, then y is part of the boundary of x at the coarser level. ($\forall x, y, z\,[y = x^{b_f} \wedge z = x^{b_c} \supset y < z]$)

As a consequence, one also gets the validity of the following statements:[14]

[14] If the topological structure at the coarser level is grounded on closed entity, these statements are also equivalent to the preceding ones.

- If x and y are separated on the coarser level, then x and y are separated on the finer level. ($\forall x, y\, [x \parallel_c y \supset x \parallel_f y]$)
- If x is self-connected on the finer level, then x is self-connected on the coarser level. ($\forall x\, [con_f(x) \supset con_c(x)]$)

The region-based approach to mereotopology thus allows for a more general relation of comparison of topological structures than point-set topology. This relation can be interpreted in terms of distinguishing between different level of granularity. Since the definition of point proposed here is relative to the mereotopological structure, points on one level of granularity need not to be points on another one.

7 Region-set approaches to 'point'

Several approaches to the definition of the notion of 'point' assume points to be sequences or sets of regions or even sets of sets of regions (cf. de Laguna 1922, Menger 1940, Whitehead 1929, Tarski 1956, Clarke 1985) and thereby introduce an additional level into the ontological structure. We will call such definitions, which are motivated by the idea that points are more abstract than regions, 'region-set' definitions. In contrast to this, the definition of 'point' given in section 4 leads to points being as concrete as regions are. This allows assumptions such as cities or other concrete objects to be points. A translation of Menger's (1940, pp. 91) definition to the nomenclature of region-based topology is presented here as an example of region-set approaches to point definitions.

Definitions:

[D37] Region x is *completely contained in* region y iff the closure of x is an inner part of y. ($x \ll_i y \equiv_{\mathrm{df}} \mathcal{R}(x) \wedge \mathcal{R}(y) \wedge x^c <_i y$)

[D38] Region x is *disjoint from* region y iff there is no region completely contained in both of them. ($x \dagger y \equiv_{\mathrm{df}} \mathcal{R}(x) \wedge \mathcal{R}(y) \wedge \neg\exists z\, [z \ll_i x \wedge z \ll_i y]$)

[D39] A sequence of regions, $x_1, x_2, \ldots$, is *strictly decreasing* iff x_{k+1} is completely contained in x_k for each k. ($\text{str-decr}((x_i)) \equiv_{\mathrm{df}} \forall k\, [x_{k+1} \ll_i x_k]$)

[D40] An *M-point* is a strictly decreasing sequence of open regions, $x_1, x_2, \ldots$, such that for any open region y which does not completely contain any of the x_k, and any open region z completely contained in y, z is disjoint from almost all x_k. ($\mathit{M\text{-}PT}((x_i)) \equiv_{\mathrm{df}} \text{str-decr}((x_i)) \wedge \forall k\, [op(x_k)] \wedge \forall y, z\, [op(y) \wedge \neg\exists k\, [x_k \ll_i y] \wedge op(z) \wedge z \ll_i y \supset \exists k\, [z \dagger x_k]]$)

[D41] An M-point, $x_1, x_2, \ldots$, is said to *lie in* region y iff y contains completely a x_k (and consequently almost all x_k).
($\mathit{M\text{-}IN}((x_i), y) \equiv_{\mathrm{df}} \mathit{M\text{-}PT}((x_i)) \wedge \mathcal{R}(y) \wedge \exists k\, [x_k \ll_i y]$)

[D42] The M-points $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are called *equal* iff each x_i contains a y_j (and consequently almost all y_j) completely and each y_i contains a x_k (and consequently almost all x_k) completely.
($(x_i) =_M (y_i) \equiv_{\mathrm{df}} \mathit{M\text{-}PT}((x_i)) \wedge \mathit{M\text{-}PT}((y_i)) \wedge \forall i\, [\exists j, k\, [y_j \ll_i x_i \wedge x_k \ll_i y_i]]$)

As Menger explicitly states, the main motivation for giving this definition is to build up an analogy between topology and arithmetic. This definition parallels a well known way of defining real numbers as sequences of rational intervals. But the parallelism is only superficial. In addition to the acceptance of set-theory, the ontological basis of defining reals in arithmetic is the existence of rationals. As a consequence, there is a canonical embedding of the rationals in the set of reals. But in the case M-points there seem to be no correlates for the rationals, and no idea of how such correlates might look.

One problem facing Menger's approach is that of identity of M-points. Menger defines identity of M-points differently from identity of sequences ([D42]). In addition, if there are two different M-points, then there are two discrete open regions such that exactly one of the points lies in either. This means that M-points are separated according to the conditions of T_2-spaces (see definition [D32]). Therefore definition [D40] restricts the range of structures which can be studied.

It seems to be more fruitful to distinguish between, on the one hand, points and other topological entities and, on the other hand, sequences of topological entities and their properties with respect to convergence. The definition proposed in section 4 allows for a rejection of the existence of points without denying the existence of M-points. The distinction between these two levels corresponds to the distinction between rational and real numbers.

In general, region-set approaches avoid the use of set theory for the discussion of regions, but introduce set theory for the definition of 'point'. The basic character of points—they do not exhibit any structure—is not reflected by these definitions. And finally, consequences of claims such as 'there are (no) points with such and such property' are in combination with this kind of definition not easy to understand.[15]

8 Conclusion

The region-based approach to mereotopology is one of the approaches interrelating topological and mereological notions in theories about regions, boundaries and connection. As we have seen, it allows for a treatment of points as entities of the same ontological level as regions. Thus, this approach gives rise to a pure topological notion of *'concrete points'* without requiring regions or boundaries to consist of points (only). The definition does not restrict the range of possibilities concerning the separation properties of points, since it allows for the existence of non-separated points. Like other mereotopological approaches, this calculus can be enriched by a definition of *'abstract points'*, which might be sets or sequences of regions, such that the interrelation of concrete and abstract points can be studied.

Assuming points and boundaries to be parts of regions permits the formalization of structural interrelations between different perspectives on a domain as discussed here with respect to the possibility of different levels of granularity or refinement. As we saw, the mereotopological approach presented generalizes the standard approach in two respects. On the one hand, an object can be a point on the coarser level while being a structured region on the finer level, which is excluded in the standard

[15] In Eschenbach (in preparation) the approach of Clarke (1985) is discussed. The main problem of this approach (reduction of topology to Classical Mereology) can be claimed to result from the difficulty of understanding such assumptions.

approach by assuming points to be on another ontological level than open sets etc. On the other hand, defining the relation of coarser and finer on the basis of the new primitive 'region' yields a more general relation than is defined in the standard approach on the basis of the primitive 'open set'.

As this discussion has shown, mereotopology is not a mere terminological variant of classical topology but may contribute to the foundations of the theory of space, time and other domains of interest in cognitive science which exhibit mereological and topological structures.

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