Linear Logics

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Outline

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Logical Calculus?

Logical Calculus?

- Formal language

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- Axioms

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- Inference rules

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Examples:

- Hilbert system, natural deduction

Logical Calculus?

- Formal language
- Axioms
- Inference rules

Examples:

- Hilbert system, natural deduction
- (resolution, tableau method)

Example: $(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\frac{(X \vee (Y \wedge Z))}{}$$

Example: $(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$Y \wedge Z$

X

$(X \vee (Y \wedge Z))$

Example: $(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\dfrac{\dfrac{(X \vee (Y \wedge Z)) \qquad \dfrac{\dfrac{X}{\quad} \quad \dfrac{X}{\quad}}{\quad} \qquad \dfrac{\dfrac{\dfrac{Y \wedge Z}{\quad}}{\quad} \quad \dfrac{\dfrac{Y \wedge Z}{\quad}}{\quad}}{\quad}}{\quad}}{\quad}$$

Example: $(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\dfrac{\dfrac{(X \vee (Y \wedge Z)) \qquad \dfrac{\dfrac{X}{X \vee Y} \quad \dfrac{X}{X \vee Z}}{} \qquad \dfrac{\dfrac{\dfrac{Y \wedge Z}{}}{} \quad \dfrac{\dfrac{Y \wedge Z}{}}{}}{}}{}}{}$$

Example: $(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$Y \wedge Z \qquad Y \wedge Z$$

$$X \qquad X$$

$$X \vee Y \qquad X \vee Z$$

$$(X \vee (Y \wedge Z)) \qquad ((X \vee Y) \wedge (X \vee Z))$$

Example: $(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\dfrac{\dfrac{(X \vee (Y \wedge Z)) \qquad \dfrac{\dfrac{X}{X \vee Y} \quad \dfrac{X}{X \vee Z}}{((X \vee Y) \wedge (X \vee Z))} \qquad \dfrac{\dfrac{\dfrac{Y \wedge Z}{Y}}{X \vee Y} \quad \dfrac{\dfrac{Y \wedge Z}{Z}}{X \vee Z}}{((X \vee Y) \wedge (X \vee Z))}}{\quad}}{\quad}$$

Example: $(X \lor (Y \land Z)) \rightarrow ((X \lor Y) \land (X \lor Z))$

$$\dfrac{(X \lor (Y \land Z)) \qquad \dfrac{\dfrac{[X]}{X \lor Y} \qquad \dfrac{[X]}{X \lor Z}}{((X \lor Y) \land (X \lor Z))} \qquad \dfrac{\dfrac{\dfrac{[Y \land Z]}{Y}}{X \lor Y} \qquad \dfrac{\dfrac{[Y \land Z]}{Z}}{X \lor Z}}{((X \lor Y) \land (X \lor Z))}}{((X \lor Y) \land (X \lor Z))}$$

Example: $(X \lor (Y \land Z)) \to ((X \lor Y) \land (X \lor Z))$

$$\dfrac{\dfrac{[(X \lor (Y \land Z))] \qquad \dfrac{\dfrac{[X]}{X \lor Y} \quad \dfrac{[X]}{X \lor Z}}{((X \lor Y) \land (X \lor Z))} \qquad \dfrac{\dfrac{\dfrac{[Y \land Z]}{Y}}{X \lor Y} \quad \dfrac{\dfrac{[Y \land Z]}{Z}}{X \lor Z}}{((X \lor Y) \land (X \lor Z))}}{((X \lor Y) \land (X \lor Z))}}{(X \lor (Y \land Z)) \to ((X \lor Y) \land (X \lor Z))}$$

Sequent Calculus

Objects aren't formulas, but *sequent*s of formulas.

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Sequent:

$$F_1, \ldots, F_n \vdash G_1, \ldots, G_m$$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$X \vdash X \qquad X \vdash X$$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\dfrac{\overline{X \vdash X}}{X \vdash X \vee Y} \qquad \dfrac{\overline{X \vdash X}}{X \vdash X \vee Z}$$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\dfrac{\dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Y} \qquad \dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Z}}{X \vdash ((X \vee Y) \wedge (X \vee Z))}$$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Y} \qquad \dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Z}}{X \vdash ((X \vee Y) \wedge (X \vee Z))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \vdash Y}}{Y \wedge Z \vdash Y}}{Y \wedge Z \vdash X \vee Y} \qquad \dfrac{\dfrac{\dfrac{}{Z \vdash Z}}{Y \wedge Z \vdash Z}}{Y \wedge Z \vdash X \vee Z}}{Y \wedge Z \vdash ((X \vee Y) \wedge (X \vee Z))}}{}$$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Y} \quad \dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Z}}{X \vdash ((X \vee Y) \wedge (X \vee Z))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \vdash Y}}{Y \wedge Z \vdash Y}}{Y \wedge Z \vdash X \vee Y} \quad \dfrac{\dfrac{\dfrac{}{Z \vdash Z}}{Y \wedge Z \vdash Z}}{Y \wedge Z \vdash X \vee Z}}{Y \wedge Z \vdash ((X \vee Y) \wedge (X \vee Z))}}{X \vee (Y \wedge Z) \vdash ((X \vee Y) \wedge (X \vee Z))}
$$

Example: $\vdash (X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Y} \quad \dfrac{\dfrac{}{X \vdash X}}{X \vdash X \vee Z}}{X \vdash ((X \vee Y) \wedge (X \vee Z))} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \vdash Y}}{Y \wedge Z \vdash Y}}{Y \wedge Z \vdash X \vee Y} \quad \dfrac{\dfrac{\dfrac{}{Z \vdash Z}}{Y \wedge Z \vdash Z}}{Y \wedge Z \vdash X \vee Z}}{Y \wedge Z \vdash ((X \vee Y) \wedge (X \vee Z))}}{\dfrac{X \vee (Y \wedge Z) \vdash ((X \vee Y) \wedge (X \vee Z))}{\vdash X \vee (Y \wedge Z) \rightarrow ((X \vee Y) \wedge (X \vee Z))}}
$$

LK - logical rules (1)

Conjunction

$$\frac{\Gamma, F \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta} \qquad \frac{\Gamma, G \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta}$$

LK - logical rules (1)

Conjunction

$$\frac{\Gamma, F \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta} \qquad \frac{\Gamma, G \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta}$$

$$\frac{\Gamma \vdash F, \Delta \quad \Gamma' \vdash G, \Delta'}{\Gamma, \Gamma' \vdash F \wedge G, \Delta, \Delta'}$$

LK - logical rules (1)

Conjunction

$$\frac{\Gamma, F \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta} \qquad \frac{\Gamma, G \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta}$$

$$\frac{\Gamma \vdash F, \Delta \quad \Gamma' \vdash G, \Delta'}{\Gamma, \Gamma' \vdash F \wedge G, \Delta, \Delta'}$$

Disjunction

$$\frac{\Gamma, F \vdash \Delta \quad \Gamma', G \vdash \Delta'}{\Gamma, \Gamma', F \vee G \vdash \Delta, \Delta'}$$

LK - logical rules (1)

Conjunction

$$\frac{\Gamma, F \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta} \qquad \frac{\Gamma, G \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta}$$

$$\frac{\Gamma \vdash F, \Delta \quad \Gamma' \vdash G, \Delta'}{\Gamma, \Gamma' \vdash F \wedge G, \Delta, \Delta'}$$

Disjunction

$$\frac{\Gamma, F \vdash \Delta \quad \Gamma', G \vdash \Delta'}{\Gamma, \Gamma', F \vee G \vdash \Delta, \Delta'}$$

$$\frac{\Gamma \vdash F, \Delta}{\Gamma \vdash F \vee G, \Delta} \qquad \frac{\Gamma \vdash G, \Delta}{\Gamma \vdash F \vee G, \Delta}$$

LK - logical rules (2)

Negation

$$\frac{\Gamma \vdash F, \Delta}{\Gamma, \neg F \vdash \Delta} \qquad \frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \neg F, \Delta}$$

LK - logical rules (2)

Negation

$$\frac{\Gamma \vdash F, \Delta}{\Gamma, \neg F \vdash \Delta} \qquad \frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \neg F, \Delta}$$

Implication

$$\frac{\Gamma, F \vdash \Delta \quad \Gamma' \vdash G, \Delta'}{\Gamma, \Gamma', F \rightarrow G \vdash \Delta, \Delta'}$$

LK - logical rules (2)

Negation

$$\frac{\Gamma \vdash F, \Delta}{\Gamma, \neg F \vdash \Delta} \qquad \frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \neg F, \Delta}$$

Implication

$$\frac{\Gamma, F \vdash \Delta \quad \Gamma' \vdash G, \Delta'}{\Gamma, \Gamma', F \rightarrow G \vdash \Delta, \Delta'}$$

$$\frac{\Gamma, F \vdash G, \Delta}{\Gamma \vdash F \rightarrow G, \Delta}$$

LK - structural rules(1)

LK - structural rules(1)

Exchange

$$\frac{\Gamma, F, G, \Theta \vdash \Delta}{\Gamma, G, F, \Theta \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, F, G, \Theta}{\Gamma \vdash \Delta, G, F, \Theta}$$

LK - structural rules(1)

Exchange

$$\frac{\Gamma, F, G, \Theta \vdash \Delta}{\Gamma, G, F, \Theta \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, F, G, \Theta}{\Gamma \vdash \Delta, G, F, \Theta}$$

Cut

$$\frac{\Gamma \vdash \Delta, D \qquad D, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

LK - structural rules(2)

Weakening

$$\frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F}$$

LK - structural rules(2)

Weakening

$$\frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F}$$

Contraction

$$\frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F}$$

Outline

Resources

Resources

D "I have a dollar."

Resources

D	"I have a dollar."
M	"I have a Mars."

Resources

D	"I have a dollar."
M	"I have a Mars."
S	"I have a Snickers."

Resources

D	"I have a dollar."
M	"I have a Mars."
S	"I have a Snickers."

"For one dollar I can get a Mars."
"For one dollar I can get a Snickers."

Resources

D	"I have a dollar."
M	"I have a Mars."
S	"I have a Snickers."

$D \rightarrow M$?	"For one dollar I can get a Mars."
$D \rightarrow S$?	"For one dollar I can get a Snickers."

Resources

D	"I have a dollar."
M	"I have a Mars."
S	"I have a Snickers."

$D \longrightarrow M$?	"For one dollar I can get a Mars."
$D \longrightarrow S$?	"For one dollar I can get a Snickers."

Wrong, because $D \longrightarrow M, D \longrightarrow S \models D \longrightarrow M \wedge S$.

conjunctions

"This paradox arises out of the confusion in classical (and intuitionistic) logic between two kinds of conjunction"[1]

conjunctions

"This paradox arises out of the confusion in classical (and intuitionistic) logic between two kinds of conjunction"[1]

Multiplicative: $A \otimes B$ ("I have both")
Additive: $A \& B$ ("I have a choice")

Connectives

Conjunctions

Multiplicative: $A \otimes B$ ("I have both")
Additive: $A \& B$ ("I have a choice")

Connectives

Conjunctions

Multiplicative: $A \otimes B$ (“I have both”)
Additive: $A \& B$ (“I have a choice”)

Disjunctions

Multiplicative: $A \parr B$ (“if not A, then B”)
Additive: $A \oplus B$ (“someone else’s choice”)

Connectives II

Implication

$A \multimap B$ ("B can be derived using A *exactly once*")

Connectives II

Implication

$A \multimap B$ ("B can be derived using A *exactly once*")

Negation

$A^{\perp}$

Exponentials

Of Course!

$!A$

Exponentials

Of Course!

$!A$

Why not?

$?A$

Example

Example

Menu: 5$

Hamburger

Coke

All-you-can-eat fries

Onion Soup or Salad
Pie or Ice Cream
(depending on availability)

Example

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Hamburger

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All-you-can-eat fries

Onion Soup or Salad
Pie or Ice Cream
(depending on availability)

$$D \otimes D \otimes D \otimes D \otimes D$$

Example

Menu: 5$

Hamburger

Coke

All-you-can-eat fries

Onion Soup or Salad
Pie or Ice Cream
(depending on availability)

$$D \otimes D \otimes D \otimes D \otimes D$$

$$\multimap$$

Example

Menu: 5$
Hamburger
Coke
All-you-can-eat fries
Onion Soup or Salad
Pie or Ice Cream
(depending on availability)

$$D \otimes D \otimes D \otimes D \otimes D$$

$$\multimap$$

$$H \otimes C \otimes !F \otimes (O \& S) \otimes (P \oplus I)$$

Outline

blub

Outline

blub

For Further Reading I

P. Lincoln
Linear Logic
ACM SIGACT News, Volume 23, Issue 2: 29-37, 1992